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SUMS CONTAINING THE FRACTIONAL PARTS OF NUMBERS

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Dedicated to the Memory of E.G. Straus and R.A. Smith

1. Introduction. Let x be a real number and $[x]$, $\{x\}$ denote respectively the integral part and the fractional part of x. Let k be a positive integer and let a be a real number. The purpose of this paper is to give an asymptotic formula for $\sum_{n\leq x} n^a\{x/n\}^k$.

Smith and Subbarao [5] obtained an asymptotic expression for this sum when $a = 0$, $k = 1$ and $n \equiv b(m)$. More recently MacLeod [4] studied it when a is an integer and k is a positive integer.

To obtain our result, we shall use a result which can be considered as an inversion formula for a class of arithmetic sums. That will be the subject of the following section.

2. Preliminaries. Let f be an arbitrary arithmetic function, arithmetic sums of the form $\sum_{n\leq x} f(n)\,[x/n]$ occur in many situations in the theory of numbers. For example, we have the well-known results

$$\sum_{n\leq x} \sigma(n) = \frac{1}{2}\sum_{n\leq x}\left(\left[\frac{x}{n}\right]^2 + \left[\frac{x}{n}\right]\right)$$

and

$$\sum_{n\leq x} \phi(n) = \frac{1}{2}\sum_{n\leq x} \mu(n)\left[\frac{x}{n}\right]^2 + \frac{1}{2},$$

where σ is the sum of the divisors of n, ϕ is Euler's totient and μ represents the Möbius function, which are used to obtain the average orders of $\sigma(n)$ and $\phi(n)$.

Let k be any non negative integer and let

$$f_k(n) = \sum_{d\mid n} g(d)\left(\frac{n}{d}\right)^k,$$

where g is any arithmetic function. Then we have

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