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PROPERTIES OF PFAFFIANS

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This paper is dedicated to E.G. Straus and R.A. Smith

1. Introduction. Several properties of Pfaffians of real skew-symmetric matrices are studied. In §2, the Pfaffian of a skew-symmetric matrix A is expressed in terms of traces of powers of A. Pfaffians of inverses and Kronecker products are studied next. If A and B are $2(2m + 1) \times 2(2m + 1)$ skew-symmetric matrices whose product is skew-symmetric, then at least one of the two matrices is singular. If A and B are $4\ell \times 4\ell$ nonsingular skew-symmetric matrices whose product is skew-symmetric, then it is shown that Pf(A), Pf(B) and Pf(AB) are either all positive all or negative. Finally, a multilinearity property of Pfaffian functions similar to the one in [4] for determinant functions is obtained, and a simple expression for the Frechet derivative of a Pfaffian function is obtained.

2. Pfaffian in terms of traces. If A is a real skew-symmetric matrix of order $2n$ (denoted by $A \in M(2n, \mathbf{R})$) then det(A) is the square of a polynomial in matrix elements of A. This polynomial is called the Pfaffian of A, and is denoted by Pf(A). It is well known, (see [3, 6, 8]) that

$$\mathrm{Pf}(A) = \sum_P \varepsilon_P a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2n-1} i_{2n}},$$

where P is the permutation

$$\begin{vmatrix} 1 & 2 \cdots & 2n \\ i_1 & i_2 & i_{2n} \end{vmatrix},$$

ε_P its sign, and the sum is taken over all permutations with the restrictions $i_1 < i_2, i_3 < i_4, \ldots, i_{2n-1} < i_{2n}, i_1 < i_3 < i_5 \cdots < i_{2n-1}$. If

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix},$$

then $\mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

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