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NUMBERS ASSOCIATED WITH STIRLING NUMBERS AND X^x

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Dedicated to the memory of my good friend E. G. Strauss

ABSTRACT We discuss two infinite triganular matrices $b(n, k)$ and $B(n, k)$ of rational integers that are associated with the matrices $s(n, k)$ and $S(n, k)$ of the Stirling numbers of the first and second kind. The numbers $b(n, k)$ were introduced in 1974 by Comtet in treating the nth derivative of x^x. They are generated by powers of the function $(1 + x)\log(1 + x)$. The numbers $B(n, k)$ are generated by powers of the inverse function.

All four matrices are treated together and numerous properties and relations are presented. In particular it is shown that $b(4h + 1, 2h) = 0$ for all integers $h > 0$. The values of the elements in a particular row of a matrix as well as the row sum when reduced modulo a prime p are also considered.

In 1974 Comtet introduced the numbers $b(n, k)$ defined by

$$\sum_{n=1}^{\infty} b(n, k)x^n/n! = \{(1 + x) \log(1 + x)\}^k/k!.$$

He used these numbers in the formula

$$\frac{d^n(x^x)}{dx^n} = x^x \sum_{j=0}^{n} (\log x)^j \binom{n}{j} \sum_{h=0}^{n-j} b(n - j, n - k - j)x^{-h}.$$

It is my purpose to show that these numbers are closely related to the Stirling numbers of the first and second kind and that they have a number of interesting properties. In fact it is important to introduce a second set of numbers $B(n, k)$ in order to treat the whole subject adequately.

We begin by introducing four infinite lower triangular matrices s, S, b, B. The elements on the nth row and kth column we denote by

$$(1) \qquad s(n, k),\ S(n, k),\ b(n, k),\ B(n, k)$$

with initial conditions

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