ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 2, Spring 1985

WELL HEIGHTS, NÉRON PAIRINGS AND V-METRICS ON CURVES

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ABSTRACT. In this paper we supplement Néron's theory of height pairings on curves by attaching to each non-constant rational function f on a curve C defined over a field K, endowed with an absolute value $| \ |_v$, a "height pairing" $\hat{\lambda}_{f,v}$ on $\mathrm{Div}(C) \times \mathrm{Div}(C)$. It is shown that the stipulation that these height pairings be "as functorial as possible" forces them to be unique (up to a constant); in particular, their restrictions to $\mathrm{Div}^\circ(C) \times \mathrm{Div}^\circ(C)$ reduce to Néron's height pairing. We also show that $\hat{\lambda}_{f,v}$ may be computed explicitly in the non-archimedean (discrete) case, via the Lichtenbaum-Shafarevich intersection theory on a suitable two-dimensional scheme over $\mathfrak{O}_v$, and in the archimedean case via Arakelov's theory of Green's functions on Riemann surfaces attached to a suitable (1, 1)-form.

0. Introduction. The theory of height pairings, which was created in 1965 by A. Néron [19] as a refinement of A. Weil's theory of distributions (Weil [23]–[25]), is important not only as a practical tool in proving diophantine statements (e.g. theorems of Mordell-Weil, Mumford, Manin etc.), but also as an intrinsic concept reflecting the finer quantitative nature of the diophantine problem in question (e.g., *Conjecture of Birch and Swinnerton-Dyer* and the recent *Theorem of Gross and Zagier*). Although Néron's theory mainly concerns abelian varieties (in fact, it is only in this case that Néron's theory completely refines Weil's theory), he does obtain (by appealing to the theory of Picard varieties) a similar (but weaker) theory of height pairings in the case of an arbitrary (smooth, complete) variety.

The purpose of this paper is to reconsider Weil's and Néron's theory of (local) heights and height pairings in the special case of curves. In doing so, we have two principal aims in mind.

The first aim is to demonstrate that it is possible to give a direct treatment of Néron's theory for curves without recourse to the theory of abelian varieties (or Jacobians). The main idea here is the observation that v-metrics on curves (as defined below in §3) yield "crude Néron

Research supported by a grant from the Deutsche Forschungsgemeinschaft
Received by the editors on February 2, 1984.