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SOME COMBINATORIAL IDENTITIES AND ARITHMETICAL APPLICATIONS

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ABSTRACT. On the strength of the Gauss-Jacobi triple-product identity the author presents a method for successively expanding infinite products

$$\prod_1^\infty (1 - x^{2n})^e(1 + ax^{2n-1})^e(1 + a^{-1}x^{2n-1})^e$$

e = 2, 4 where $a, x \in \mathbf{C}$, $a \neq 0$ and $|x| < 1$. Some arithmetical applications are noted.

1. Introduction. The mainspring of our discussion is the Gauss-Jacobi triple-product identity

$$\prod_1^\infty (1 - x^{2n})(1 + ax^{2n-1})(1 + a^{-1}x^{2n-1}) = \sum_{-\infty}^{\infty} x^{n^2}a^n \tag{1}$$

valid for each pair of complex number a, x such that $a \neq 0$ and $|x| < 1$. For a proof see [**3**, p. 282]. Our objective is the presentation of an elementary method for successively raising this identity to the second and fourth powers. To this end, we recall that, for an arbitrary pair of integers k, n, with $k \geqq 2$ and $n \geqq 0$, $r_k(n)$ denotes the cardinality of the set

$$\{(x_1, \ldots, x_k) \in \mathbf{Z}^k \mid n = x_1^2 + \cdots + x_k^2\}.$$

Our results can now be stated as

THEOREM 1. *For each pair of complex numbers a, x such that $a \neq 0$ and $|x| < 1$,*

$$\begin{aligned}\prod_1^\infty &(1 - x^{2n})^2(1 + ax^{2n-1})^2(1 + a^{-1}x^{2n-1})^2 \\ &= \sum_{-\infty}^{\infty} x^{2n^2} \sum_{-\infty}^{\infty} x^{2n^2}a^{2n} + x\sum_{-\infty}^{\infty} x^{2n(n+1)} \sum_{-\infty}^{\infty} x^{2n(n+1)}a^{2n+1}\end{aligned} \tag{2}$$

and

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