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PROBLEMS AND SOME RESULTS CONCERNING THE DIOPHANTINE EQUATION $1 + A + A^2 + \cdots + A^{x-1} = P^y$.

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Dedicated to the memory of E. G. Straus and R. A. Smith

We assume throughout the paper that a, x, p and y are positive integers, that $a > 1$ and that p is an odd prime. The primary purpose of the paper is to raise several as yet unsettled questions concerning the equation of the title and to try to induce people to solve these problems.

As a first, and perhaps not very important, example we notice that $1 + 3 + 3^2 + 3^3 + 3^4 = 11^2$. In the time-honoured spirit of Erdös, we offer \$25 for any other such example in which a is an odd prime and in which $y > 1$.

It has been known for quite some time (see, for instance, [1] that x must be prime, say $x = Q$, and that $Q = \text{ord}_p a$. (Here $\text{ord}_m a$ is defined to be the least t such that $a^t \equiv 1 \pmod{m}$.) We now specialize by assuming that "a" is from Southern California*, i.e., $a = r^b$, where r is an odd prime. In this case it is possible to prove that $b = Q^\lambda$ for some $\lambda \geqq 0$, as the following argument shows:

Suppose that there exists a prime $R \neq Q$ such that $R|b$ and write $b = RB$. Then

$$p^y = \frac{a^x - 1}{a - 1} = \frac{(r^b)^Q - 1}{r^b - 1} = \frac{(r^{RB})^Q - 1}{r^{RB} - 1} = \frac{[(r^B)^Q]^R - 1^R}{(r^B)^R - 1^R}$$
$$= \frac{((r^B)^Q - 1)}{(r^B - 1)} \cdot \frac{\{1 + (r^B)^Q + ((r^B)^Q)^2 + \cdots + ((r^B)^Q)^{R-1}\}}{\{1 + r^B + (r^B)^2 + \cdots + (r^B)^{R-1}\}}$$
$$= \frac{(r^B)^Q - 1}{r^B - 1} \cdot \frac{\Phi_R((r^B)^Q)}{\Phi_R(r^B)} = \frac{(r^B)^Q - 1}{r^B - 1} \cdot \Phi_{QR}(r^B),$$

where $\Phi_n(x)$ denotes the nth cyclotomic polynomial (see, for instance, [2]). Thus

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* (This terminology has been adopted in honour of the research done on the title equation and related matters by Dennis Estes, Bob Guralnick, Murray Schacher and Ernst Straus. The current paper owes a lot to the work of these people.)