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DENSITY OF M-TUPLES WITH RESPECT TO POLYNOMIALS

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1. Introduction. Let h, k and m be positive integers with $m \geqq 2$, x real and $\geqq 1$, and $f_1, f_2, \ldots, f_m$ arbitrary nonconstant polynomials with integer coefficients. Let $M(x; f_1, f_2, \ldots, f_m; h; k)$ denote the number of m-tuples $\langle x_1, x_2, \ldots, x_m \rangle$ of positive integers such that $x_i \leqq x$ for $1 \leqq i \leqq m$ and $(f_1(x_1), f_2(x_2), \ldots, f_m(x_m))_k = h$. Here the symbol $(a_1, a_2, \ldots, a_m)_k$ stands for the greatest k-th power common divisor of $a_1, a_2, \ldots, a_m$ with the convention that $(0, 0, \ldots, 0)_k = 0$. We also write $d(f_1, f_2, \ldots, f_m; h; k) = \lim_{x\to\infty} x^{-m} M(x; f_1, f_2, \ldots, f_m; h; k)$ and call this the density of the m-tuples $\langle x_1, x_2, \ldots, x_m \rangle$ with $(f_1(x_1), f_2(x_2), \ldots, f_m(x_m))_k = h$. In the special case when $f_1(x) = f_2(x) = \ldots, f_m(x) = x$, $h = k = 1$, it is known due to Césaro [2], J. J. Sylvester [7], D. N. Lehmer [5] and J. E. Nymann [6] that this density is $1/\zeta(m)$, $\zeta(s)$ being Riemann's ζ-function. Recently, R. N. Buttsworth [1] determined this density in the general case with $k = 1$ but his proof contains some serious errors—for example, his lemma 2.2 basic to his work is fallacious (See Section 4). In this paper we evaluate this density in the following cases: (1) $k = 1$ and at least one of the polynomials is of degree less than m (2) $k \geqq 2$ and at least one of the polynomials is linear and (3) $k \geqq 1$ and the polynomials are primitive and irreducible. In fact, in each of the cases we obtain an asymptotic formula for $M(x; f_1, f_2, \ldots, f_m; h; k)$ with an 0-estimate for the error term (see Theorems 1, 2, and 3.)

2. Preliminaries. We denote by $\rho_i(n)$ the number of solutions mod n of the congruence

$$f_i(x) \equiv 0 (\text{mod } n),\ \rho(n) = \prod_{i=1}^{m} \rho_i(n),\ \rho_i^*(n) = \max\{1, \rho_i(n)\}$$

and

$$\rho^*(n) = \prod_{i=1}^{m} \rho_i^*(n),\ D_i = \deg f_i(x),\ D = \prod_{i=1}^{m} D_i \text{ and } u = \min_{1\leqq i\leqq m} D_i.$$

Also $\mu(n)$ denotes the Möbius function and $\omega(n)$ the number of distinct prime factors of n.

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