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METRIC TRANSFORMATIONS OF THE REAL LINE

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1. A metric transformation between two metric (or semi-metric) spaces M_1 and M_2 is defined to be a function f such that for some function $\rho: \mathbf{R}^+ \to \mathbf{R}^+$, called the scale function associated with f, $\rho(d_1(x, y)) = d_2(f(x), f(y))$, where $x, y \in M$. The set $f(M_1)$ is said to be a metric transform of M_1. In this paper all metric transforms from the real line in Euclidean n-space are characterized.

The notion of a metric transformation was introduced by Wilson **[10]** in 1935. In 1938 von-Neumann and Schoenberg **[8]** characterized all continuous metric transformations of the real line, $\mathbf{R}$, into Hilbert space. This powerful result shows that the scale functions ρ corresponding to such transformations are those, and only those, functions which satisfy the condition

$$\rho^2(t) = \left(\int_0^\infty \frac{\sin^2 tu}{u^2}\, d\alpha(u)\right),$$

where α is non-decreasing and $\int_1^\infty u^{-2} d\alpha(u) < \infty$. They also showed that, in order that $f(\mathbf{R})$ be embeddable in $\mathbf{E}^n$ (finite dimensional Hilbert space), it is necessary and sufficient that α increase at only a finite number of points. In this case

$$\rho^2(t) = \sum_1^m A_i^2 \sin^2 k_i t + c^2 t^2,$$

and in a suitable coordinate system,

(1) $\quad f(t) = (A_1 \cos k_1 t, A_1 \sin k_1 t, \ldots, A_m \cos k_m t, A_m \sin k_m t, \mathrm{ct})$

If $f(\mathbf{R})$ is embeddable in E^n, but not in E^{n-1}, then, for n odd, $2m = n - 1$ and $c \neq 0$, while $2m = n$ and $c = 0$ for n even. As a helix is typical, von-Neumann and Schoenberg refer to continuous metric transforms of $\mathbf{R}$ as screw curves.

Metric transformations, including the von-Neumann and Schoenberg result, have appeared in the literature of late in connection with a method of data analysis known as Multidimensional Scaling. (See **[1]**, **[3]**, **[6]** and **[7]**). Here one takes a semi-metric space M_1 and some other metric

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