ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 15, Number 1, Winter 1985

THE GENERA OF PSL(F_q)-LÜROTH COVERINGS

ARTHUR K. WAYMAN

1. Introduction. In [3] H. Hasse studies the ramification theory of Kummer and Artin-Schreier cyclic coverings of an algebraic function field in one variable. These cyclic extensions are special cases of a wider class of function fields which we will entitle Lüroth coverings. In this paper we will study in detail the ramification theory of PSL($\mathbf{F}_q$)-Lüroth coverings. We will classify all genus zero and genus one PSL($\mathbf{F}_q$)-Lüroth coverings of a rational function field and construct bases for the spaces of differentials of the first kind for coverings with genus $\geqq 2$.

For notation, definitions, and standard theorems used here, the reader may consult the bibliography.

2. Lüroth coverings. Let k be a field and Y an indeterminate over k. Denote by PGL(k) the group of k-automorphisms of the rational function field $k(Y)$. For each element $\sigma \in \mathrm{PGL}(k)$ there are elements a_σ, b_σ, c_σ, $d_\sigma \in k$ with $a_\sigma d_\sigma - b_\sigma c_\sigma \neq 0$ satisfying $\sigma(f) = f((a_\sigma Y + b_\sigma)/(c_\sigma Y + d_\sigma))$ for all $f \in k(Y)$. We recall that two substitutions

$$Y \to \frac{aY+b}{cY+d} \text{ and } Y \to \frac{a'Y+b'}{c'Y+d'}$$

induce the same k-automorphism of $k(Y)$ if and only if $(a', b', c', d') = (\lambda a, \lambda b, \lambda c, \lambda d)$ for some $\lambda \in k^x = k - \{0\}$.

Let $\mathscr{G}$ be a finite non-trivial subgroup of PGL(k). If $k(Y)^{\mathscr{G}}$ is the subfield of $k(Y)$ left invariant by the action of $\mathscr{G}$, then $k(Y)^{\mathscr{G}}$ contains k and from galois theory we have $[k(Y): k(Y)^{\mathscr{G}}] = |\mathscr{G}|$, where $|\mathscr{G}|$ denotes the cardinality of $\mathscr{G}$. By Lüroth's theorem (see van der Waerden [5]) there is an element $Z_{\mathscr{G}}$ in $k(Y)$ such that $k(Y)^{\mathscr{G}} = k(Z_{\mathscr{G}})$. We can write $Z_{\mathscr{G}} = U_{\mathscr{G}}/V_{\mathscr{G}}$ for some $U_{\mathscr{G}}$, $V_{\mathscr{G}} \in k[Y]$ with $(U_{\mathscr{G}}, V_{\mathscr{G}}) = 1$. Moreover,

$$\deg_Y Z_{\mathscr{G}} = \max\{\deg_Y U_{\mathscr{G}}, \deg_Y V_{\mathscr{G}}\} = |\mathscr{G}|.$$

We remark that any other generator of $k(Y)^{\mathscr{G}}$ is of the form $(aZ_{\mathscr{G}} + b)/(cZ_{\mathscr{G}} + d)$ where $a, b, c, d \in k$ and $ad - bc \neq 0$.

Let K be an algebraic function field in one variable over the algebraically

Received by the editors on December 20, 1982 and in revised form on September 27, 1983.