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THE THEORY OF FORCED, CONVEX, AUTONOMOUS, TWO POINT BOUNDARY VALUE PROBLEMS

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1. Introduction. We consider the semilinear, autonomous, forced, two point boundary value problem with a parameter λ,

$$u'' + bu' + \lambda f(u) = 0, \tag{1.1a}$$

$$\begin{aligned} \alpha_0 u(0) - \alpha_0' u'(0) &= \gamma_0 \\ \alpha_1 u(1) + \alpha_1' u'(1) &= \gamma_1. \end{aligned} \tag{1.1b}$$

Here f is a positive, convex function on the closed half-line $[0, +\infty)$, b is a real number, and α_i, α_i', and γ_i ($i = 0$ and 1) are nonnegative real numbers which satisfy additional conditions described in §2.

We give an almost complete description of the positive solutions of (1.1), with essentially no assumptions on f other than positivity and convexity, describing in detail how the structure of the solution set depends on the asymptotic behavior of the function f. It is well known that, at least for sufficiently smooth f, there exists a positive number λ^* such that (1.1) has solutions for positive $\lambda < \lambda^*$ and has no solutions for $\lambda > \lambda^*$. We show, using Leray-Schauder degree theory, that (1.1) has at most two positive solutions for each value of $\lambda > 0$ except for the special case described below. From known results, we deduce necessary and sufficient conditions on f for the existence of two solutions for certain values of λ and describe the values of λ for which two solutions exist.

The results presented here go beyond results already in the literature in several ways. First, the calculation of the fixed point index of the solutions and the proof of the existence of at most two solutions for all $\lambda \neq \lambda^*$; second, the conversion of the problem with a nonmonotonic convex nonlinearity to a problem with an isotone convex nonlinearity; third, the consideration of functions which are not strictly convex, and hence the possibility, for certain boundary conditions, of infinitely many solutions for $\lambda = \lambda^*$; finally, the lack of smoothness and monotonicity of f, with the convexity assumption only and the possibility that $f_+'(0) = -\infty$. These results show specifically how the results for the linear problem

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