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A NEW PROOF OF THE CWIKEL-LIEB-ROSENBLJUM BOUND

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1. Introduction. Consider the operator $-\Delta + V$ acting on $L^2(\mathbf{R}^3)$, where $V(x)$ is a potential in $L^{3/2}(\mathbf{R}^3)$. Let $N(V)$ be the dimension of the spectral projection of $-\Delta + V$ on $(-\infty, 0]$. Then it is known **[1, 5, 8]** that

$$N(V) \leqq C \int_{\mathbf{R}^3} |V_-(x)|^{3/3}\, dx, \tag{1.1}$$

which C is a constant and V_- denotes the negative part of V. The inequality (1.1) was derived in three quite different ways by Lieb **[5]**, Cwikel **[1]** and Rosenbljum **[8]**. The best value for the constant C was obtained by Lieb **[5]** and is $C = .116$. Attempts have been made to obtain the best constant C but the results are rather inconclusive **[3]**. However, it is known **[5]** that $C \geqq .0780$.

Here we obtain a new derivation of (1.1) with constant $C = .168$. Our approach is adapted from Lieb's method **[6, 7]** to show that Dirac's semi-classical formula for exchange energy **[2]** bounds the quantum exchange energy. In fact we merely paraphrase the arguments of **[7]** so that (1.1) may be regarded as a corollary of the exchange energy bound.

Another new proof of (1.1) has also recently been given by Li and Yau **[4]**. It is quite different from the one presented here as well as the three previous derivations. Despite the claim in **[4]** the constant obtained there is three times worse than Lieb's value of .116.

We turn to our proof of (1.1). As is standard in all approaches to this problem, we consider a different problem, which is equivalent by the Birman-Schwinger principle **[9]**. Thus we assume $V(x) \leqq 0$, for all $x \in \mathbf{R}^3$, and put $V(x) = -W(x)^2$, where $W(x) \geqq 0$ for $x \in \mathbf{R}^3$. We consider the operator A on $L^2(\mathbf{R}^3)$ with integral kernel

$$a(x, y) = W(x)W(y)\,[4\pi|x - y|]^{-1}. \tag{1.2}$$

Since $W \in L^3(\mathbf{R}^3)$ with norm $\| W\|_3$ it follows that A is a positive Hilbert-Schmidt operator and thus has discrete spectrum $\mu_1 \geqq \mu_2 \geqq \cdots \geqq 0$. Then to prove (1.1) we need to show that for any $\lambda > 0$,

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