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KUMMER CONGRUENCES IN FORMAL GROUPS AND ALGEBRAIC GROUPS OF DIMENSION ONE

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0. Introduction. Kummer congruences for the Bernoulli numbers and the coefficients in the expansion of the secant function were discovered by E.E. Kummer around 1850. Subsequently much work has been done to show similar types of congruences for the coefficients of various generating functions.

L. Carlitz was the first to treat Kummer congruences for "Hurwitz series" systematically. We briefly outline the history and definitions here.

Let k be an algebraic number field, i.e., a finite degree extension of $\mathbf{Q}$, the field of rational integers. Let R be an integral domain in k containing $\mathbf{Z}$, the ring of rational integers. (Normally R will be 0_k, the ring of integers of k, or perhaps a subring of $0_k[1/h]$ for some nonzero rational integer h. The important point here is that almost all rational primes are not units so that congruences mod pR are not trivial except for finitely many primes.)

We define a Hurwitz series over R as a power series $f(t)$ of the form

$$f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \text{ where } a_n \in R.$$

If $a_0 = 0$ and $a_1 = 1$, then $f(t)$ has an inverse $\lambda(t)$, i.e., $f(\lambda(t)) = t = \lambda(f(t))$, of the form $\lambda(t) = \sum_{n=0}^{\infty} e_n t^n/n!$ where $e_0 = 0$, $e_1 = 1$, and $e_n \in R$ for all $n \geqq 0$. We consider only those $f(t)$ with $a_0 = 0$, $a_1 = 1$ satisfying the Hypothesis: For all $n \geqq 1$, $(n-1)!|e_n$, i.e., the inverse $\lambda(t) = \sum_{n=1}^{\infty} \varepsilon_n t^n/n$ where $\varepsilon_n \in R$.

This hypothesis is equivalent to the "integrality condition" that the formal derivative

$$f'(t) = \sum_{\nu=0}^{\infty} d_\nu f^\nu \text{ with } d_\nu \in R.$$

(The d_ν are a priori in k.) In particular if there exists a nonzero polynomial $P(X, Y) \in k[X, Y]$ such that $P(f, f') = 0$, then f satisfies the hypothesis for an appropriate choice of R, cf. e.g., [7].

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