ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

SEMIALGEBRAIC BOREL-MOORE-HOMOLOGY

HANS DELFS

Dedicated to the memory of Gus Efroymson

Borel and Moore developed an homology theory for locally compact topological spaces using sheaf theoretical methods ([2], see also [3]). It allows us to consider not only homology with compact support (which is the same as classical singular homology) but also homology with arbitrary support. This general concept of homology has important applications. A striking example is the proof that every algebraic variety over **R** (resp. over **C**) has a fundamental class [1]. To obtain similar results for varieties over an arbitrary real closed field R, or over its algebraic closure $C = R(\sqrt{-1})$, we are forced to introduce a semialgebraic version of Borel-Moore-homology. In this paper we will sketch how this may be done. (In [4] and [6] we handled the special case of complete support). Semialgebraic Borel-Moore-homology may even be of interest in the case $R = \mathbf{R}$, because it is far simpler and more elementary than the classical sheaf theoretical theory.

R denotes a real closed field. Let M be an affine semialgebraic space over R [5] and A be a semialgebraic subset of M. We can triangulate M, A simultaneously, i.e., there exists a finite geometric simplicial complex X over R, a subcomplex Y of X and a semialgebraic isomorphism $\varphi: X \xrightarrow{\sim} M$ which maps Y onto A [6, §2]. Notice that the simplicial complex X is not complete in general. Some faces may be missing. Adding these missing faces we obtain a simplicial complex $\bar{X}$, the closure of X. $\bar{X}$ is a simplicial complex in the classical sense and a complete semialgebraic space.

The basic building blocks in classical algebraic topology are closed simplices. But not all simplices occuring in X are closed. Nevertheless, they seem to contribute to the homology of M. So it is quite natural to take the open simplices as the building blocks of a homology theory for semialgebraic spaces. This idea will be formalized in the following.

Definition 1. An abstract simplicial complex K is a pair $(E(K), S(K))$ consisting of a set $E(K)$, called the vertices of K, and a set $S(K)$ of non-empty finite subsets of $E(K)$, called the simplices of K, such that $E(K) = \bigcup(s \mid s \in S(K))$. K is called closed if, in addition, every non-empty subset t of a simplex $s \in S(K)$ is also a simplex.