ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

MINIMAL GENERATION OF BASIC SEMIALGEBRAIC SETS

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Dedicated to the Memory of Gus Efroymson

Introduction. Let R be a real closed field and V an affine algebraic R-variety. We assume that $V(R)$ is Zariski-dense in V. A basic semialgebraic set $S \subset V(R)$ is a set of the form $S = S(f_1, \ldots, f_m) := \{x \in V(R) \mid f_i(x) > 0,\ i = 1, \ldots, m\}$ for suitable $f_i \in R[V]$. How many f_i are needed for such a representation of S? It is shown that there exists a finite upper bound depending only on the dimension n of V. This bound is equal to n for $n \leqq 3$. I did not succeed in proving (or disproving) this for $n > 3$. Anyway, the best bound is $\geqq n$. We shall also characterize the basic semialgebraic sets among the open semialgebraic sets.

1. The real spectrum. For a quasicompact scheme S we denote by $(X(S), \beta(S))$ the real spectrum [4]. This is a restricted topological space $X(S)$ with base $\beta(S)$ [2]. For an R-variety V one has also the restricted topological space $(V(R), \gamma(V))$, where $\gamma(V)$ is the lattice generated by all sets, which are basic semialgebraic after restriction to open affine subsets of V. By the ultrafilter theorem [2] one has canonical isomorphisms

$$(\hat{V}(R), \hat{\gamma}(V)) \underset{p}{\overset{f}{\rightleftarrows}} (X(V), \beta(V))$$

where $\wedge$ means canonical ultrafilter completion of a restricted topological space.

Now let $x_1, \ldots, x_l$ be real points of the R-Variety V, and let A be the semilocal ring $A = \lim_{\rightarrow} \mathcal{O}(U)$, U open in V, $x_1, \ldots, x_l \in U$. We set $\hat{V}(x_1, \ldots, x_l) := \{F \in \hat{V}(R) \mid x(F)$ generalizes some $x_i\}$, and provide this with the induced base $\hat{\gamma}(x_1, \ldots, x_l)$. Then the projection λ: $\mathrm{Spec}(A) \rightarrow V$ defines an imbedding

$$X(\lambda)\colon (X(\mathrm{Spec}(A)), \beta(\mathrm{Spec}(A)) \rightarrow (X(V), \beta(V)),$$

moreover, one has the commutative diagram

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