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ON PYTHAGOREAN REAL IRREDUCIBLE ALGEBROID CURVES

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Dedicated to the Memory of Gus Efroymson

In this note we deal with the pythagoras number p of certain 1-dimensional rings, i.e., real irreducible algebroid curves over a real closed ground field k. The problem we are concerned with is to characterize those real irreducible algebroid curves which are pythagorean (i.e., $p = 1$). We obtain two theorems involving the value-semigroup. Then we apply them to solve the cases of: (a) Gorenstein curves, (b) planar curves, (c) monomial curves, and (d) curves of multiplicity $\leqq 5$. Finally, two conjectures are stated.

1. Statement of the theorems. Let k be a fixed real closed field. A real irreducible algebroid curve is any real 1-dimensional complete local integral domain A whose residual field is k.

Let pA denote the pythagoras number of A (i.e., the least $p \geqq 0$ such that any sum of squares is a sum of p squares). It can be shown that pA is finite. When $pA = 1$, A is called pythagorean.

Now we recall some definitions [1]. As is known, the derived normal ring $\bar{A}$ of A is a discrete valuation ring and we denote by ν its valuation. The semigroup $\Gamma = \nu(A - \{0\})$ is called value-semigroup of A. Then

i) The multiplicity of A = least positive integer m in Γ.

ii) The degree of the conductor of A in $\bar{A}$ = least positive integer $c \in \Gamma$ such that each $n \geqq c$ is in Γ.

Conversely, if Γ is a numerical semigroup (i.e., $\Gamma \subset N$ and $N - \Gamma$ is finite) the right sides above give definitions of m and c. Finally we denote by $\mathscr{M}_\Gamma$ the class of all curves whose value-semigroup is Γ and by $\mathscr{P}yth_\Gamma$ the class of all pythagorean curves in $\mathscr{M}_\Gamma$. Then we have

THEOREM I. *$\mathscr{P}yth_\Gamma \neq \phi$ if and only if for each $q \in \Gamma$ the set $\Gamma_q = \{p - q \mid p \geqq q, p \in \Gamma\}$ is a semigroup.*

Now set $d = \min\{p \in \Gamma \mid p \not\equiv 0(\overline{m})\} \cup \{c\}$ and $E = \{p \in \Gamma \mid p \geqq d\}$. Then