ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

EXTENDED ARTIN-SCHREIER THEORY OF FIELDS

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Dedicated to the memory of Gus Efroymson

Introduction. This survey is concerned with recent developments in the Artin-Schreier theory of fields. The basic notion of an ordering of a field has been extended to the more general notion of an ordering of higher level. This extension has opened the way to a natural, far-ranging extension of the ordinary Artin-Schreier theory. Between 1924 and 1927, the foundation of the Artin-Schreier theory was laid by two papers of E. Artin ([**1**], [**2**]), by two joint papers of E. Artin and O. Schreier ([**3**], [**4**]) and by R. Baer's contribution [**5**]. As an introduction to this survey we recall some main features of these papers and the investigations they inspired.

It is thown that a field K can be ordered if and only if it is formally real which, by definition, means that -1 is not a sum of squares in K. Moreover, an element of K is proved to be a sum of squares if and only if it is contained in all orderings of K.

The maximal real agebraic extensions R of K (the real closures of K) are shown to admit the unique ordering R^2. Via $R \mapsto R^2 \cap K$ ($R^2 \cap K$ is an ordering in K), their conjugacy classes over K correspond bijectively to the orderings of K.

Starting from these results Artin was able to solve Hilbert's 17^{th} problem in the affirmative. Artin's proof related, for the first time, the theory of real fields with real algebraic geometry. This becomes especially clear in S. Lang's version of this proof [**35**] which in turn gave rise to the Real Nullstellensatz by Dubois [**27**] and Risler [**40**]. An up-to-date account of this relationship can be found in several papers contained in [**23**].

Real closed fields are characterized by the property that their algebraic closure is a finite, nontrivial extension or, equivalently, by the property that their absolute Galois group is a nontrivial finite group, in fact a cyclic group of order 2. From this point of view, the theory of real closed fields contributes to the question of characterizing which profinite groups may occur as the absolute Galois group of a field. It is this problem which is of interest for this paper.