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HOVANSKY' THEOREM AND COMPLEXITY THEORY.

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Dedicated to the memory of Gus Efroymson

The additive complexity of $P \in R[X_1, \ldots, X_n]$ is related to the set of zeros of P in R^n.

1. Hovansky's theorems. (Cf. [2], [3]). The results of Hovansky are in the spirit of Bezout's theorem, but in the real case. Let us recall Descartes's lemma.

LEMMA 1.1. *If* $P = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, *the number of positive real roots of* P *is smaller than the number of changes of signs in the sequence* $a_0, \ldots, a_n$.

PROOF. This is very simple by induction on n, using Rolle's theorem.

COROLLARY 1.2. *The number of positive real roots of* P *is smaller than the number of non-zero monomials in* P.

The result of Hovansky is a generalisation of this-corollary.

THEOREM 1.3. *Let* $F_1, \ldots, F_n \in R[X_1, \ldots, X_n, Y_1, \ldots, Y_k]$, $\deg.F_i = m_i$, *where* $Y_i = e^{\langle a^i, X\rangle}(1 \leqq i \leqq k)$ *with* $\langle a^i, X\rangle = \sum_{j=1}^n a_j^i X_j$, $a_j^i \in R$. *Then the number of non-degenerate roots in* R^n *of the system* $\{F_i(X, Y(X)) = 0, 1 \leqq i \leqq n\}$ *(with* $X = (X_1, \ldots, X_n)$*) is* $\leqq 2^{(1/2)k(k-1)}(1 + \sum m_i)^k \prod m_i$.

The proof is by induction on k, beginning with the classical Bezout theorem, and using an old method of Liouville to "kill" the exponentials, and a variant of Rolle's theorem.

COROLLARY 1.4. *Let* $P_1, \ldots, P_n \in R[X_1, \ldots, X_n]$, *the total number of monomials in* $(P_1, \ldots, P_n)$ *being* k; *then the number of non-degenerate solutions in* R_+^n *of the system* $P_1 = \cdots = P_n = 0$, *is* $\leqq (1 + n)^k 2^{k(k-1)/2}$.

PROOF. Put $X_i = e^{Y_i}$, and use Theorem 1.3.

REMARKS 1.5. a) Probably the bound in the corollary can be greatly improved.

b) Theorem 1.3 can be generalised to a large set of analytic functions.

2. Additive complexity of polynomials in one variable over R. The additive