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SEMI-INTEGRAL EXTENSIONS AND PROPER MORPHISMS

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Dedicated to the memory of Gus Efroymson.

Semi-integral extensions were introduced by Brumfiel in [1] from an algebraic point of view, and applied to study algebraic varieties over real closed fields. In this note we develop the connection between semi-integral extensions and proper morphisms.

Let R be a real closed field. If X is an irreducible algebraic variety over R, we denote by X_c the maximum dimension locus of X, by $R[X]$ the coordinate ring of X and by $R(X)$ its field of rational functions; we denote by β_d the ordering in $R[X]$ induced by the weakest ordering in $R(X)$. A dominant morphism $\varphi: X \to Y$ of irreducible algebraic varieties over R is called semi-integral if $(R[X], \beta_d)$ is a semi-integral extension of $(R[Y], \beta_d \cap R[Y])$ via φ. Then we prove the following proposition.

PROPOSITION 1. *The following are equivalent.*
(a) *φ is semi-integral*
(b) *For every bounded semialgebraic set $S \subset Y$, $[\varphi^{-1}(S)] \cap X_c$ is bounded.*
(c) (i) *For every closed semialgebraic set $S \subset X_c$, $\varphi(S)$ is closed, and,* (ii) *For every $y \in Y$, $[\varphi^{-1}(y)] \cap X_c$ is bounded.*

From this proposition we can deduce a result of Brumfiel, with no need of Hironaka's desingularization theorem [2].

COROLLARY 2. *Let f, $g \in R[X]$, Then f/g is semi-integral over $R[X]$ if and only if f/g is bounded on bounded semialgebraic sets of $X_c - V(g)$.*

Condition (c) in proposition 1 can be taken as a definition of real properness. Then, in terms of real spectra [3], we obtain a valuative Criterion for properness analogous to the classical one. Note that $\tilde{X} = \mathrm{Spec}_R R[X]$, $\tilde{Y} = \mathrm{Spec}_R R[Y]$, and $\tilde{\varphi}: \tilde{X} \to \tilde{Y}$ the morphism induced by φ. We have

PROPOSITION 2. *The following are equivalent.*
(a) (i) *For every closed constructible set $S \subset \tilde{X}$, $\tilde{\varphi}(S)$ is closed and*
(ii) *For every rational point $y \in Y$, $\tilde{\varphi}^{-1}(y)$ is real complete.*
(b) (Valuative Criterion) *For every real closed field F, every real valuation*