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REAL FIELDS WITH SMALL GALOIS GROUPS

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Dedicated to the memory of Gus Efroymson

In his survey in this volume, Becker mentions how his search for real fields with simple absolute Galois groups led him to the discovery of hereditarily pythagorean fields. This note will illustrate his remark. It will serve two purposes: Provide a method of constructing examples and point out directions for further classification of real fields. Proofs of the results will appear elsewhere.

Let k be a real field, Ω its algebraic closure and $G = \mathrm{Gal}(\Omega|k)$, its absolute Galois group. G can be very large and so one looks for simple identifiable types of small closed subgroups H of G containing involutions so that the corresponding fixed fields of H are all real.

THEOREM. *Let R be a real closed field and $\alpha \in R$. Let k be a subfield of R such that $\alpha \notin k$ and maximal with respect to these two properties. Then we have exactly one of the following*:

1) *k is a Geyer field with $G \cong \mathbf{Z}_p \times \mathbf{Z}/2\mathbf{Z}$ where p is a prime integer.*

2) *The commutator subgroup G' is the free pro-2-group on an infinity of generators and $G/G' \cong \mathbf{Z}_p \times \mathbf{Z}/2\mathbf{Z}$ where p is a prime integer. Also G has two generators σ and τ where σ is an involution.*

For explicit examples one may take $\alpha = 2^{1/3}$ in the field R of real algebraic numbers. The resulting k is a Geyer field with $G = \mathbf{Z}_3 \times \mathbf{Z}/2\mathbf{Z}$. For the second case, take C to be the cyclotomic extension of $\mathbf{Q}$ obtained by adjoining all the p^n th roots of unity for all $n \geqq 1$, p being an odd prime. Then $\mathrm{Gal}(C|\mathbf{Q}) = \mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$. Then there is a subexension $C_1|\mathbf{Q}$ with $\mathrm{Gal}(C_1|\mathbf{Q}) \cong \mathbf{Z}_p$. Then C_1 contains a Galois extension $\mathbf{Q}(\alpha)|\mathbf{Q}$ of degree p. If we now consider a subfield k of R maximal with respect to the exclusion of α, then $\mathrm{Gal}(\Omega|k)$ falls under case 2.

The Theorem seems important for classifying real fields. For example, if a field k is not hereditarily pythagorean, then its absolute Galois group G contains a subgroup of Case 2 of the Theorem. It therefore leads to the following problem: characterize those real fields k, the commutator