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WITT RINGS AND K-THEORY

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Dedicated to the memory of Gus Efroymson

ADDED IN PROOF. A version of this article was distributed by the author at the AMS-NATO Summer conference on Ordered Fields and Real Algebraic Geometry, held at Boulder, Colorado, July 3–9, 1983. A few months later, N. Schwartz discovered counterexamples to some of the Propositions. It turned out that what was needed was a better definition of the ring $C(X)$ of §3. N. Schwartz provided such a definition. Also, H. Delfs and M. Coste helped clarify the matter.

Here is the correction needed. In the notation of §3, an element of $C(X)$ should be a constructible continuous section $s: X \to X_{A[T]}$ such that the image $s(X)$ is relatively closed in $\pi^{-1}(X) \subset X_{A[T]}$. In fact, without this closedness condition, Proposition 3.3 is false in general, since clearly 3.3 implies $s(X)$ is closed. With the better definition, Proposition 3.7 and 3.9, as stated below, can also be improved. Specifically, Proposition 3.7 is true for any ring homomorphism $\gamma: A \to B$, as hoped for in Remark 3.8. Also, Proposition 3.9 is true for any integral extension $\gamma: A \to B$, so $C(X_A) \to C(X_{A/I})$ is surjective for any ideal $I \subset A$.

1. Introduction. The primary purpose of these notes is to outline a proof of the following result.

THEOREM. *Let A be any commutative ring, X_A its real spectrum. Then there is a natural ring homomorphism $W(A) \to KO(X_A)$ with both kernel and cokernel 2-torsion groups.*

Here, $W(A)$ is the Witt ring of A, and $KO(X_A)$ is something like the real K-theory of a topological space. Much of the paper is devoted to the definition and properties of $KO(X)$ for any constructible subset $X \subseteq X_A$. In fact, four "definitions" are given, all isomorphic, and each a direct analogue of a construction of the classical real K-theory of a compact (= quasi-compact and Hausdorff topological) space X. More about these four constructions below.

Here are some special cases and applications of the theorem. First, if $A = k$ is a field, then it turns out that $KO(X_k) = \text{Cont}(X_k, \mathbf{Z})$, the ring of

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