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A NOTE ON FIXED-POINT CONTINUED FRACTIONS AND AITKEN'S Δ^2-METHOD

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ABSTRACT. Limit periodic continued fractions can be accelerated, and, in some instances, analytically extended by the use of certain modifying factors. This procedure is actually Aitken's Δ^2-method when applied to equivalent continued fractions/power series. Both acceleration and continuation results are given.

The continued fraction

(1) $$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots, \qquad \text{with complex } \{a_n\} \text{ and } \{b_n\},$$

is called limit periodic if $a_n \to a$ and $b_n \to b$.

In accordance with the following procedure, (1) may be conceptualized as a composition of linear fractional transformations:

Let $t_n(w) = a_n/(b_n + w)$ for $n = 1, 2, \ldots$, and set $T_1(w) = t_1(w)$, $T_n(w) = T_{n-1}(t_n(w))$ for $n = 2, 3, \ldots$. Then $a_1/b_1 + \cdots + a_n/(b_n + w) = T_n(w)$, and, in particular, the nth approximant of (1) equals $T_n(0)$.

It is usually the case that each t_n has two distinct fixed points, α_n and β_n. When $|\alpha_n| < |\beta_n|$, α_n is called the attractive fixed point and β_n, the repulsive fixed point of t_n. If we assume (1) is limit periodic, then ordinarily $t_n(w) \to t(w) = a/(b + z)$ with $\alpha_n \to \alpha$, $\beta_n \to \beta$. Each t_n can be written $t_n(w) = \alpha_n\beta_n/[(\alpha_n + \beta_n) - w]$, so that (1) can be recast in fixed-point form

(2) $$\frac{\alpha_1\beta_1}{\alpha_1 + \beta_1} - \frac{\alpha_2\beta_2}{\alpha_2 + \beta_2} - \cdots$$

Over the last ten years several papers have appeared describing and investigating a simple modification of (1) that frequently accelerates convergence and may analytically continue the function represented by (1) (if $a_n = a_n(z)$, $b_n = b_n(z)$) into a larger domain. See, e.g., [**2**], [**3**], [**7**], [**8**]. The modification requires the use of $T_n(\mu_n)$ in lieu of $T_n(0)$. μ_n customarily takes on fixed point values α, β, α_{n+1} or β_{n+1}. Motivation for this technique comes from the study of infinite iterations of a single linear

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