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A CHARACTERIZATION OF ORIENTED GRASSMANN MANIFOLDS

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Introduction. Let $G_{p,q}$ denote the oriented Grassmann manifold of p-planes in $\mathbf{R}^{p+q}$. Our purpose is to give a characterization of $G_{p,q}$ and its non-compact dual $G^*_{p,q}$ in terms of a parallel tensor field T satisfying certain algebraic conditions and its behaviour on geodesic spheres. When $q = 1$ our result contains that of L. Vanhecke and T. J. Willmore on spaces of constant curvature ([5]. [2]). For $q = 2$, a different characterization has been obtained by B. J. Papantoniou using the Hermitian structure which exists for that case [4].

In the course of our work we give (Proposition 3) an algebraic characterization of the tensor T on a vector space V^{pq}. Although every Riemannian manifold trivially carries a parallel tensor field satisfying the given conditions, namely $T(X, Y, Z) = g(Y, Z)X$, for $p, q \geqq 2$, T plays a significant role in the geometry of the Grassmann manifolds, somewhat analogous to the underlying almost complex structure on a Kähler manifold. In [5] Vanhecke and Willmore have also characterized the complex space forms in terms of their Kähler structures and the shape of their geodesic spheres. They have similarly characterized the remaining rank 1 symmetric spaces.

Some Properties of $\mathbf{G}_{p,q}$. We consider $G_{p,q}$ as the Riemannian symmetric space $SO(p+q)/(SO(p) \times SO(q))$. Then following Kobayashi and Nomizu [3 pp. 271–273], for example, we may identify the tangent space at a point $m \in G_{p,q}$ with the vector space of real $p \times q$ matrices. Moreover the inner product

$$g(X, X) = \operatorname{tr} XX^t \tag{1}$$

at m gives rise to an invariant metric g on $G_{p,q}$ with curvature tensor R at m given by

$$R(X, Y)Z = XY^tZ + ZY^tX - ZX^tY - YX^tZ. \tag{2}$$

Similarly for the non-compact dual $G^*_{p,q}$ of $G_{p,q}$ the curvature at a point is given by the negative of this expression. Any other invariant metric

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