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HERMITEAN QUADRICS AS CONTACT MANIFOLDS

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ABSTRACT. The notions of real and complex contact manifolds are classical and it is evident that the projective cotangent bundle of real projective space, a real contact manifold, is a real form of the projective contangent bundle of complex projective space, a complex contact manifold. Hermitean quadrics, which are real manifolds in complex projective space, are real contact manifolds and are also real forms of the projective cotangent bundle of complex projective space. The latter is not evident and it is the purpose of this paper to establish these assertions, exhibit their connection with the anti-polarities of classical projective geometry, and to show that these two types of real contact manifolds constitute all of the real forms of the projective cotangent bundle of complex projective space, as homogeneous contact manifolds. The development of these facts leads to the observation that Hermitean quadrics are principal circle bundles over products of complex projective spaces and generalize the Hopf bundle as real contact manifolds.

1. Introduction. A real contact manifold is a $(2n-1)$-dimensional manifold with a contact structure given by a global maximal rank Pfaffian form [2]. The standard examples are odd-dimensional spheres and projective cotangent bundles of real n-dimensional manifolds. Boothby and Wang have shown how real contact manifolds arise as principal circle bundles over Kähler manifolds, the Pfaffian form being obtained from the Kähler form [2]. A well-known example is the projective cotangent bundle $M^{(2n-1)}$ of real projective space $P^{(n)}$; it is a principal circle bundle over a complex quadric Q^{n-1} in complex projective space P^n.

More interesting, and less known, are the Hermitean quadrics $\Phi_s^{(2n-1)}$, of signature s, in P^n. These are real contact manifolds, and are principal circle bundles over the products $P^s \times P^{n-s-1}$ of complex projective spaces. The contact manifolds $M^{(2n-1)}$ and $\Phi_s^{(2n-1)}$ are distinct since, for example, Q^{n-1} and $P^s \times P^{n-s-1}$ have different second Betti numbers.

However: The distinct real contact manifolds $M^{(2n-1)}$ and $\Phi_s^{(2n-1)}$ are, in fact, real forms of the same complex contact manifold, namely the projective contangent bundle M^{2n-1} of complex projective space P^n.

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