ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 3, Summer 1984

# PRODUCTS OF TWO ABELIAN SUBGROUPS

BERNHARD AMBERG

Every group $G = AB$, which is the product of two abelian subgroups $A$ and $B$, is metabelian by a well-known result of Itô [4]. In this short note some further statements on the structure of such groups are given. For instance, the center, the $FC$-center, the hypercenter and the $FC$-hypercenter of $G$ are 'factorized' as products of a subgroup of $A$ and a subgroup of $B$ (Theorem 2.2). The Fitting subgroup and the Hirsch-Plotkin radical of $G$ are in general not factorized in a corresponding way. However, some sufficient conditions are given, under which these important characteristic subgroups are factorized (Theorems 2.4 and 2.5). It is also shown that if $G$ is not cyclic of prime order and if $A \neq G$ or $B \neq G$, then there is at least one factorized normal subgroup $N$ of $G = AB$ with $1 \neq N \neq G$ (Theorem 3.1).

The notation is standard; see for instance [8] and [9].

**1. The factorizer.** The following result of Wielandt [12] is useful for the investigation of factorized groups.

LEMMA 1.1. *If the group $G = AB$ is the product of two subgroups $A$ and $B$, then the following conditions of the subgroup $S$ of $G$ are equivalent*:

(a) $S = (A \cap S)(B \cap S)$ *and* $A \cap B \subseteqq S$,

(b) *If* $ab \in S$ *with* $a \in A$ *and* $b \in B$, *then* $a \in S$.

A subgroup $S$ of the factorized group $G = AB$ which satisfies the equivalent conditions of Lemma 1.1 is called *factorized.*

Since intersections of arbitrary many factorized subgroups of $G = AB$ are factorized subgroups of $G$, every normal subgroup $N$ of $G$ is contained in a smallest factorized subgroups $X = X(N)$ of $G$, which we call the *factorizer* of $N$ in $G$. By [1], Theorem 1.7, p. 108, the following holds.

*Lemma* 1.2. *If the group $G = AB$ is the product of two subgroups $A$ and $B$ and if $N$ is a normal subgroup of $G$, then*

$$X = X(N) = AN \cap BN = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN).$$

This implies the following result.

Received by the editors December 3, 1982 and in revised form May 18, 1983.