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CAMPBELL'S CONJECTURE ON A MAJORIZATION-SUBORDINATION RESULT FOR CONVEX FUNCTIONS

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Let S denote the set of all normalized analytic univalent functions f, $f(z) = z + \cdots$, in the open unit disc U. Let f, F, and w be analytic in $|z| < r$. We say that f is majorized by F, $f \ll F$, in $|z| < r$, if $|f(z)| \leqq |F(z)|$ in $|z| < r$. We say that f is subordinate to F, $f \prec F$, in $|z| < r$ if $f(z) = F(w(z))$ where $|w(z)| \leqq |z|$ in $|z| < r$.

Majorization-subordination theory begins with Biernacki who showed in 1936 that if $f'(0) \geqq 0$ and $f \prec F (F \in S)$, in U, then $f \ll F$ in $|z| < 1/4$. In the succeeding years Goluzin, Tao Shah, Lewandowski and MacGregor examined various related problems (for greater detail see [1]).

In 1951 Goluzin showed that if $f'(0) \geqq 0$ and $f \prec F (F \in S)$ then $f' \ll F'$ in $|z| < 0.12$. He conjectured that majorization would always occur for $|z| < 3 - \sqrt{8}$ and this was proved by Tao Shah in 1958.

In a series of papers [1, 2, 3], D. Campbell extended a number of the results to the class $\mathscr{U}_\alpha$ of all normalized locally univalent ($f'(z) \neq 0$) analytic functions in U with order $\leqq \alpha$ where $\mathscr{U}_1 = K$ is the class of convex functions in S. In particular in [3] he showed that if $f'(0) \geqq 0$ and $f \prec F(F \in \mathscr{U}_\alpha)$ then $f' \ll F'$ in $|z| < \alpha + 1 - (\alpha^2 + 2\alpha)^{1/2}$ for $1.65 \leqq \alpha < \infty$ where $\alpha = 2$ yields $3 - \sqrt{8}$. Note that $\alpha = 1$ yields $2 - \sqrt{3}$, the radius of convexity for S. Campbell's proof breaks down for $1 \leqq \alpha < 1.65$ because of two different bounds being used for the Schwarz function with different ranges of α. Nevertheless, he conjectured that the result is true for all $\alpha \geqq 1$.

In this paper we combine a subordination result of Ruscheweyh's, some variational techniques and some tedious computations to verify the conjecture for $\alpha = 1$, i.e., we show that if $f'(0) \geqq 0$ and $f \prec F(F \in K)$ in U then $f' \ll F'$ for $|z| \leqq 2 - \sqrt{3}$. We note that our method of proof relies on the convexity of F in a number of places so that it is unlikely that it would extend to larger α's.

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