

ON HAAR MEASURE OF CERTAIN HYPERCOMPLEX UNITARY GROUPS

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ABSTRACT. By using the theory of hypercomplex matrices some results of Toyama [5] on Haar measure of orthogonal and unitary groups are extended to similar results for Haar measure on the unitary groups of hypercomplex matrices. A formula for Haar measure of unitary and unitary symplectic hypercomplex matrix groups of the division algebras is derived, in terms of the Cayley parametrization of these matrices.

1. Introduction. In his celebrated paper on the theory of invariants, Hurwitz [2] introduced the notion of invariant measures on group manifolds. He gave an explicit expression for Haar measure of unitary unimodular groups, and also for orthogonal groups by using the theory of generalized polar coordinates. Later Weyl [6], p. 169, and p. 217] obtained other expressions for the same measures. Toyama [5] obtained different expressions for these measures by using Cayley's parametric representations of orthogonal and unitary matrices. Toyama's [5] results are as follows.

THEOREM 1. *The infinitesimal volume element $d\Omega$ of Haar measure on the unitary group of $n \times n$ unitary matrices Ω is*

$$(1) \quad d\Omega = |I + H^2|^{-n} dH.$$

Here H is a $n \times n$ Hermitian matrix, $H = (h_{ij})$, $dH = \pi_{i < j} dh_{ij} = \pi_{i < j} da_{ij} \pi_{i < j} db_{ij}$, $H = A + iB$, $A = (a_{ij})$, $B = (b_{ij})$, A is symmetric and B is skewsymmetric, and $n \times n \Omega$ is represented by Cayley's parametric representation as

$$(2) \quad \Omega = (I + iH)(I - iH)^{-1}.$$

Note that Toyama [5] considers only the group of those unitary matrices which can be represented by (2).

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