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CONVEX POLYTOPES AND RETRACTIONS OF ABELIAN GROUPS

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Introduction. For any group G, let $F(G)$ denote the semigroup of finite non-empty subsets of G. A semigroup homomorphism σ: $F(G) \to G$ satisfying $\sigma(\{g\}) = g$ for all g in G is called a *retraction* of G. The notion of a group admitting a retraction generalizes the notion of a lattice-ordered group because in any lattice-ordered group the mapping $A \to \wedge A$ is a retraction (cf., [**1**]). This example of a retraction induced by a lattice order has the property that the effect of the mapping on $F(G)$ is determined uniquely by its effect on two element subsets. This is not so for all retractions, and [**1**, example 6.1], gives an instance where two distinct retractions agree on all two element subsets. The question of whether distinct retractions can agree on sets of cardinality less than or equal to n for arbitrary n is dealt with in this paper.

Also, in looking at a retraction σ on a group G, the notion which corresponds to that of an l-subgroup is the notion of a σ-subgroup—a subgroup H of G such that σ restricted to $F(G)$ is a retraction of H. In this paper we also deal with the question of whether a subgroup H of G with the property that all sets in $F(H)$ of cardinality less than n get mapped by σ to H must necessarily be a σ-subgroup.

Our approach considers only retractions of divisible abelian groups and builds on observations made in [**3**] and [**4**]. In the process of studying retractions we get a correspondence between retractions and homomorphisms from a semigroup of convex polytopes in Q^n to Q^n, so some of our results are essentially geometric in nature.

I. Retractions and convex polytopes. Throughout, G will be a torsion free divisible abelian group, hence a rational vector speace. For convenience we take G to be of finite rank.

If σ is any retraction of G, and A, B, C are sets satisfying $A + C = B + C$, then $\sigma(A) = \sigma(B)$. Hence for A, B in $F(G)$, we define $A \sim B$ if there is a C in $F(G)$ with $A + C = B + C$. The following proposition is then easy to verify.

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