A BRIEF SURVEY AND HISTORY OF
ASYMPTOTIC PRIME DIVISORS

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ABSTRACT. Recently there has been considerable interest in high
powers of ideals, their integral closures, and the associated primes
of these ideals, and many of these results are closely related to
other areas of current research interest. The purpose of this paper
is to summarize these new results and to give a brief historical
overview of their development.

0. Introduction. Powers of ideals have played an important role in com-
mutative algebra. (For example, in completions, quadratic transfor-
mations, associated graded rings, Rees rings, asymptotic closure and
integral closure of ideals, Hilbert function and multiplicity, etc.) Recently,
quite a few results have shown that large powers of ideals and their prime
divisors are of considerable interest in their own right and are also closely
connected to other areas of current research interest, such as going-down
between prime ideals and the catenary chain conjectures. In this paper
a brief survey of this area together with a sketch of the historical an-
tecedents of several of the results is given.

§1 contains a review of the terminology and a summary of the notation
used in this paper. §2 is mainly historical; it contains a summary of
various generalizations of a 1916 theorem of Macaulay. §3 contains some
additional (but less closely related) generalizations of Macaulay's result.
The first few of these are quite new and yield characterizations of Cohen-
Macaulay rings and of unmixed and quasi-unmixed Noetherian rings, and
the others are very useful older generalizations. §4 contains numerous
results on the prime divisors of I^n and of $(I^n)_a$ with n large. Included are
characterizations of these sets of prime ideals, characterizations of various
kinds of rings in terms of these sets, and results showing when a given
prime ideal P is a prime divisor of all large powers of all ideals of a certain
type contained in P. Several results concerning the analytic spread of an