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DISTRIBUTIVE, MODULAR AND SEPARATING ELEMENTS IN LATTICES

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Given the importance of distributive lattices as a class, it was a natural step to consider distributivity of elements in an arbitrary lattice L. For instance an element d is called *distributive* if $d \vee (x \wedge y) = (d \vee x) \wedge (d \vee y)$ for all $x, y \in L$, and *separating* if $d \vee x = d \vee y$ and $d \wedge x = d \wedge y$ to gether imply $x = y$. An important early result was that in a modular lattice any distributive (or dually distributive) element is in fact *neutral*, that is, distributive, dually distributive and separating. A deeper result is that the same is true in weakly modular lattices [**3**, §III. 2].

In this paper the above result is extended in other directions, notably in M-symmetric and "Θ- modular" lattice. To do this we introduce some notions which might be considered as "modularity" of elements in a fashion similar to that for "distributivity" above. For instance an element d of L is *left* [*right*] *modular* if $d\ M\ a$ [$a\ M\ d$] for all $a \in L$, and *weakly separating* if $d \vee x = d \vee y$, $d \wedge x = d \wedge y$ and $x \leqq y$ together imply $x = y$. Such elements do indeed arise (in a nontrivial manner) in congruence lattices, for example.

The first main result proved is that in an M-symmetric lattice, any element which is both distributive and dually distributive is neutral. Given the lack of duality inherent in M-symmetry, this is perhaps the strongest result that might be expected. On the other hand it is shown that in an M-symmetric algebraic lattice satisfying DCC, any dually distributive element is neutral. Counterexamples show that these results cannot be extended.

In the final section the concept of "Θ-modularity", introduced (in a rather special context) by Spitznagel [**7**], is considered and its relationship with the earlier concepts is demonstrated. Roughly speaking, given an equivalence Θ on the lattice L, L is Θ-modular if each of its elements weakly separates each Θ-class. (Thus a modular lattice is Θ-modular for any Θ). The case in which we are most interested corresponds to the equivalence $\Theta_d = \{(a, b) \in L \times L : a \vee d = b \vee d\}$, when d is an arbitrary element of L. Our main result here is that if d is distributive (so that Θ_d is in fact a congruence on L) and if L is Θ_d-modular, then d must be neutral.

In a sequel [**5**] these results will be used in a discussion of congruence

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