ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 2, Spring 1983

LINEAR TRANSFORMATIONS PRESERVING SETS OF RANKS

LEROY B. BEASLEY

ABSTRACT. Let T be a linear transformation on $M_{m,n}(F)$, the set of all $m \times n$ matrices over the algebraically closed field F, and let R_j denote the subset of all matrices of rank j. Further let $R_E = \bigcup_{j \in E} R_j$ where E is a subset of $\{0, 1, \ldots, \min(m, n)\}$. We explore the structure of T when $T(R_E) \subseteqq R_E$.

1. Introduction. Let $M_{m,n}(F)$ denote the set of all $m \times n$ matrices over the algebraically closed field F and let $\rho(A)$ denote the rank of the matrix A. Let R_j denote the set of all matrices $A \in M_{m,n}(F)$ such that $\rho(A) = j$. If E is a subset of $\{0, 1, \ldots, \min(m, n)\}$, let $R_E = \bigcup_{j \in E} R_j$. In this notation consider the following problem: if $T: M_{m,n}(F) \to M_{m,n}(F)$ is a linear transformation, $E \subseteqq \{0, 1, \ldots, \min(m, n)\}$, and $T(R_E) \subseteqq R_E$, then what is the structure of T? There are two trivial cases: $E = \{0, 1, \ldots, \min(m, n)\}$ and $E = \{1, \ldots, \min(m, n)\}$. In the first case T need only be linear and in the second case T need only be linear and nonsingular.

Throughout the remainder of the paper we will assume that T is a linear transformation on $M_{m,n}(F)$ and that $m = \min(m, n)$.

Some research has been done for the case $E = \{k\}$ [**1**, **6**, **7**] and, in fact, in each known case when E is a proper subset of $\{1, \ldots, m\}$ the structure of T is the same [**1**, **2**, **3**, **6**, **7**]. We demonstrate that structure in the following theorem of Marcus, Moyls and Westwick [**6**, **7**].

THEOREM 1. *If $T(R_1) \subseteqq R_1$, then there exist $m \times m$ and $n \times n$ nonsingular matrices U and V respectively such that either*

i) *$T: A \to UAV$ for all $A \in M_{m,n}(F)$*

or

ii) *$m = n$ and $T: A \to UA^tV$ for all $A \in M_{m,n}(F)$ where A^t denotes the transpose of A.*

For easy reference we define a transformation T satisfying (i) or (ii) in Theorem 1 as a rank-1-preserver. We note that as a consequence of [**2**, Thm. 4] we have the following theorem.

THEOREM 2. *If E is a subset of $\{0, 1, \ldots, m\}$, $E \neq \{1, 2, \ldots, m\}$, and if T is nonsingular, then T is a rank-1-preserver.*

Received by the editors on August 4, 1980, and in revised form on November 2, 1981.