ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 2, Spring 1983

# A PRESERVATION OF INTEGRABILITY CHARACTERIZATION THEOREM

WILLIAM D.L. APPLING

ABSTRACT. Suppose $N$ is a positive integer and $\mathbf{Q}$ denotes the set to which $g$ belongs if and only if $g$ is a function from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ such that for some $(w_1, \ldots, w_N)$ in $\mathbf{R}^N$ and $d > 0$, $g(w_1, \ldots, w_N)$ is bounded on $[-d; d]$. A characterization is given of those elements $f$ of $\mathbf{Q}$ having the property that if $U$ is a set, $\mathbf{F}$ is a field of subsets of $U$, each of $\alpha_1, \ldots \alpha_N$ is a function from $\mathbf{F}$ into a collection of subsets of $\mathbf{R}$ with bounded union, $\xi$ is a real-valued, bounded finitely additive function defined on $\mathbf{F}$ and each of the set function integrals $\int_U \alpha_1(I)\xi(I), \ldots, \int_U \alpha_N(I)\xi(I)$ exists, then the integral $\int_U f(\alpha_1(I), \ldots, \alpha_N(I), \xi(I))$ exists, these integrals being limits for subdivision refinement.

**1. Introduction.** Suppose $N$ is a positive integer. In a previous paper [5] (see [2] for the earlier interval function version) the author showed the following preservation of integrability characterization theorem (see §2 for the notion of integral.

THEOREM 1.A.1. *If $f$ is a function from $\mathbf{R}^N$ into $\mathbf{R}$, then the following two statements are equivalent.*

1) *If $\mathbf{F}$ is a field of subsets of a set $U$, $\xi$ is a real-valued bounded finitely additive function defined on $\mathbf{F}$, and each of $\alpha_1, \ldots, \alpha_N$ is a function from $\mathbf{F}$ into a collection of subsets of $\mathbf{R}$ with bounded union* (in [5] *the $\alpha$'s were single valued, but the argument carries over for this version with trivial modifications*) *such that each of the integrals $\int_U \alpha_1(I)\xi(I), \ldots, \int_U \alpha_N(I)\xi(I)$ exists, then the integral $\int_U f(\alpha_1(I), \ldots, \alpha_N(I))\xi(I)$ exists.*

2) *The function $f$ is continuous.*

In this paper we extend the above theorem. Notice that if $f$ is given as above and $h$ is a function from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ such that for each $(x_1, \ldots, x_N, z)$ in $\mathbf{R}^{N+1}$, $h(x_1, \ldots, x_N, z) = f(x_1, \ldots, x_N)z$, then the conclusion of statement 1) above has the form "$\int_U h(\alpha_1(I), \ldots, \alpha_N(I), \xi(I))$ exists". The question naturally arises as to whether there exists a class, $Q$, of functions from $\mathbf{R}^{N+1}$ into $\mathbf{R}$ that includes the functions of the form "$f(x_1, \ldots, x_N)z$", and a subset which has the integrability

*AMS (MOS) subject classifications* (1970): Primary 28A25; Secondary 26A54.
*Key words and phrases*: Set function intgral, preservation of integrability characterization.
Received by the editors on February 7, 1979, and in revised form on May 3, 1982.