ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 2, Spring 1983

A PAIR OF BIORTHOGONAL SETS OF POLYNOMIALS

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ABSTRACT. The two sets of polynomials $\{J_n^{(\alpha,\beta)}(x)\}$ and $\{K_n^{(\alpha,\beta)}(x)\}$ where $J_n^{(\alpha,\beta)}(x)$ is of degree n in x^k and $K_n^{(\alpha,\beta)}(x)$ is of degree n in x $(n = 0, 1, 2, \ldots)$ are constructed so that they are biorthogonal on (0, 1) with respect to discrete distribution $d\Omega(\alpha, \beta; x)$ which has jumps $[\alpha q]_\infty[\beta q]_i(\alpha q)^i/[\alpha\beta q^2]_\infty[q]_i$ at $x = q^i$. When $k = 1$ these reduce to the little q-Jacobi polynomials. Various other properties are also given.

1. Introduction. Let $\alpha(x)$ be a "distribution" on $[a, b]$ (finite or infinite), that is, $\alpha(x)$ is bounded, incrasing on (a, b), with infinitely many points of increase, and such that $\int_a^b x^n \, d\alpha(x) < \infty$ for all $n \geqq 0$.

The set of polynomials $\{P_n(x)\}$, and the set of polynomials $\{Q_n(x)\}$, deg $Q_n(x) = n$ for $n = 0, 1, 2, \ldots$ are said to be biorthogonal on (a, b) with respect to $d\alpha(x)$ if

$$\int_a^b P_n(x)Q_m(x)d\alpha(x) = h_n\delta_{n,m} \tag{1.1}$$

with $h_n \neq 0$ and δ_{nm} the familiar Kronecker delta. In this paper we shall take $P_n(x)$ to be of degree n in x^k where k is fixed.

Didon [7] and Deruyts [6] considered this concept in some detail. For example, given the set $\{P_n(x)\}$ the set $\{Q_n(x)\}$ is uniquely determined and conversely.

This concept has been reconsidered in [11], [12]. It is shown that (1.1) is equivalent to (1.2) and (1.3),

$$\int_a^b x^i P_n(x)d\alpha(x) \begin{cases} = 0 & 0 \leqq i < n, \\ \neq 0 & i = n \end{cases} \tag{1.2}$$

and

$$\int_a^b x^{ik} Q_n(x)d\alpha(x) \begin{cases} = 0 & 0 \leqq i < n, \\ \neq 0 & i = n. \end{cases} \tag{1.3}$$

Thus if $k = 1$, $\{P_n(x)\}$ and $\{Q_n(x)\}$ collapse to the set of orthogonal polynomials associated with $\alpha(x)$ on (a, b). Both Didon and Deruyts gave as an example the case $d\alpha(x) = x^{\alpha-1}(1 - x)^{\beta-1} \, dx$ on (0, 1). More

Received by the editors on November 13, 1981, and in revised form on January 8, 1982.