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TYPICALLY-REAL FUNCTIONS OF ORDER p

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1. Introduction. A number of authors have considered generalizations of Rogosinski's class T [9] of typically-real functions. Primary attention has been given to the class of functions $f \in H(\mathbf{B})$ (i.e., holomorphic in the unit disk, $\mathbf{B}$) which have real coefficients in their Taylor expansions about zero and, for some positive integer p, satisfy one of the following two conditions:

(I) $f \in H(\bar{\mathbf{B}})$ and $\operatorname{Im} f$ changes sign exactly $2p$ times as z traverses $\partial\mathbf{B}$, and

(II) $\exists \rho \in (0, 1)$ such that $\operatorname{Im} f$ changes sign exactly $2p$ times as z traverses $|z| = r$ for each $r \in (\rho, 1)$.

Functions satisfying (I) or (II) have at most p zeros in $\mathbf{B}$, counting multiplicity. Robertson [7] showed that members, f, of this class which have a zero of order p at $z = 0$ and normalization $f(z) = z^p + \cdots$ may be represented in the form

$$f(z) = \frac{z}{1 - z^2} \prod_{k=1}^{p-1} \frac{z}{1 - 2z \cos s_k + z^2}\, u(z)\,, \tag{1}$$

where $s_k \in \mathbf{R}$ (the real numbers), $k = 1, \ldots, p - 1$, and $u \in H(\mathbf{B})$, $\operatorname{Re} u > 0$. Extremal problems for the class have been studied by various authors, including Goodman, Robertson, and Umezawa. In particular, the coefficient problem is treated in [3], [4], [10].

Conditions (I) and (II), however, are not completely satisfactory for defining a class of functions to be called typically-real of order p. There are functions of the form (1) which satisfy neither (I) nor (II). Moreover, there are sequences in this class which converge (uniformly on compact subsets of $\mathbf{B}$) to limits which do not satisfy (I) or (II). Examples are given in §5 of this paper.

In §2 we define an argument function for the boundary values of suitably restricted members of $H(\mathbf{B})$. We use this boundary argument to formulate a less restrictive condition than (I) or (II) which generates a class, $T(p)$, of functions which we call typically-real of order p. Functions in $T(p)$ will be required to have exactly p zeros, counting multiplicity. We show that $T(p)$ is characterized by a product representation like (1) which accounts for zeros other than $z = 0$. Furthermore, $T(p)$ is essentially

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