ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 1, Winter 1983

LOCALIZATION WITH RESPECT TO A CLASS OF SPACES

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1. Introduction. Let $\mathscr{F}$ be a set of compact Hausdorff spaces. Given a Hausdorff space X, we construct the universal example of a map $X \to Y$ so that every map $\varDelta \to Y$, $\varDelta \in \mathscr{F}$, is null-homotopic. For suitably chosen $\mathscr{F}$, this localization is shown to be the Quillen plus construction, if X is a CW-complex with $[\pi_1(X), \pi_1(X)]$ perfect, and the Sullivan localization, if X is a nilpotent CW-complex with the Malcev-Lazard completion of $\pi_1(X)$ equal to 0.

2. $\mathscr{F}$-localization. Throughout this paper, we will assume all spaces pointed, maps basepoint preserving and homotopies relative to the basepoint. Cones, suspensions and mapping cylinders and cones will all be reduced. Let $\mathscr{F}$ be any set of spaces. We say that a space X is *$\mathscr{F}$-local* if every map f: $\varDelta \to X$, $\varDelta \in \mathscr{F}$, is null-homotopic. An *$\mathscr{F}$-localization* of a space X is a map $L_{\mathscr{F}}: X \to X_{\mathscr{F}}$ so that

(i) $X_{\mathscr{F}}$ is $\mathscr{F}$-local and

(ii) if Y is $\mathscr{F}$-local and $g: X \to Y$, then there is a map $h: X_{\mathscr{F}} \to Y$ so that $h \circ L_{\mathscr{F}} \cong g$.

Given X, define $F_{\mathscr{F}}(X)$ (or just $F(X)$ if $\mathscr{F}$ is understood) to be the space obtained from X by adjoining the mapping cones of all maps $\varDelta \to X$, $\varDelta \in \mathscr{F}$. Note that $X \subset F_{\mathscr{F}}(X)$. Define $F^0_{\mathscr{F}}(X) = X, F^n_{\mathscr{F}}(X) = F_{\mathscr{F}}(F^{n-1}_{\mathscr{F}}(X))$ and let $X_{\mathscr{F}} = \varinjlim F^n_{\mathscr{F}}(X)$; define $L_{\mathscr{F}}: X \to X_{\mathscr{F}}$ to be the obvious inclusion.

THEOREM 2.1. *If X is Hausdorff and each space in $\mathscr{F}$ is compact Hausdorff, then $L_{\mathscr{F}}: X \to X_{\mathscr{F}}$ is an $\mathscr{F}$-localization.*

PROOF. Let $\varDelta \in \mathscr{F}, f: \varDelta \to X_{\mathscr{F}}$. Since $\varDelta$ is compact, there is an integer n so that $f(\varDelta) \subset F^n_{\mathscr{F}}(X)$. Therefore, f is null-homotopic in $F^{n+1}_{\mathscr{F}}(X)$ and so in $X_{\mathscr{F}}$.

Suppose $g: X \to Y$ where Y is $\mathscr{F}$-local. Clearly, g extends to a map $F_{\mathscr{F}}(X) \to F_{\mathscr{F}}(Y)$, and so to a map $g_{\mathscr{F}}: X_{\mathscr{F}} \to Y_{\mathscr{F}}$. But $L_{\mathscr{F}}(Y): Y \to Y_{\mathscr{F}}$ is a homotopy equivalence, and letting ϕ be a homotopy inverse, $h = \phi \circ g_{\mathscr{F}}$ satisfies the relation $h \circ L_{\mathscr{F}}(X) \cong g$.

AMS (MOR) subject classification: 57D99.
Received by the editors on July 7, 1978.