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DIOPHANTINE CHAINS

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ABSTRACT. We solve a large class of quadratic binary Diophantine equations without resort to the theory of quadratic number fields. Examples are given of equations amenable to our approach, including some which are intractable by classical methods. A corollary is drawn concerning the size of smallest possible solutions to certain quadratic forms.

In most treatments of Diophantine equations, the determination of all integral solutions to the quadratic binary equation

$$(*)\qquad ax^2 + bxy + cy^2 + dx + ey + f = 0$$

with integral coefficients is reduced to the solution of a Pellian equation $x^2 - Dy^2 = K$, where $D = b^2 - 4ac$. To solve the latter one must find the fundamental unit of the real quadratic field $Q(\sqrt{D})$. This is a laborious process, to be avoided if possible. In [4] we presented an alternative approach, first suggested by W. H. Mills [2], which applies whenever $a \neq 0$, $b \neq 0$, $c \neq 0$, a divides b and d, and c divides b and e. In this paper we improve and simplify the methods of [4] and show how they may also be used in many cases when (*) does not immediately satisfy the divisibility conditions. There are still many equations we cannot solve (e.g., $x^2 - 13y^2 = 1$, whose smallest solution is $x = 649$, $y = 180$), and we do not hold much hope that the method can ever be made completely general. However, it does apply to most examples we have seen considered in number theory texts and, in these cases, effects a substantial savings in computation. It has the added attraction of being completely elementary; in particular, no knowledge of the theory of quadratic number fields is required.

If the pair (x, y) satisfies $(*)$, then so do the pairs (x', y) and (x, y') where $ax' = -ax - by - d$ and $cy' = -cy - bx - e$. The divisibility conditions are needed to ensure that x' and y' will be integers whenever x and y are. Thus, each integral solution (x, y) generates an endless (but possibly cyclic) chain $\cdots x''y'xyx'y'' \cdots$ and in [4] we showed that the number of such

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