ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 1, Winter 1983

REARRANGEMENTS OF DIVERGENT SERIES

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ABSTRACT. Let $\sum a_k$ be a divergent series of positive numbers. The rate of divergence of $\sum a_k$ is related to the behavior of subseries and to rearrangements of the series. We show the rate of divergence of $\sum a_k$ is determined by the convergent subseries of $\sum a_k$ and also show that the rate of divergence can be changed, through rearrangement, to give some other predesignated rate of divergence.

1. Introduction. Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive numbers with $a_k \to 0$ and $\sum a_k = \infty$. In this paper we consider the rate of divergence of the partial sums $A_n = \sum_1^n a_k$ as $n \to \infty$. We prove some results concerning the rate of growth of these partial sums and how it may be altered through rearrangement of the series.

Our first results, making up the second section of the paper, show that the rate of growth of A_n as $n \to \infty$ is determined by the convergent subseries of $\sum a_k$. These results are, in a sense, "inverse" to previous results of Banerjee and Lahiri [1] and Salat [6]. (These papers consider the sums of convergent subseries of divergent series and how "often" a particular positive P can be the sum of a convergent subseries).

The third section of the paper concerns arrangements of $\sum a_k$. Let π: $\mathbf{Z}^+ \to \mathbf{Z}^+$ be a permutation of the positive integers. In [2], [3] and [4], Diananda found conditions under which $\sum_1^n a_k$ and $\sum_1^n a_{\pi(k)}$ are asytomtic as $n \to \infty$. In [7], Stenberg also considered rearrangements of divergent series and studied the divergent subseries of the rearrangements. Our main result in the third section considers another aspect of the rearrangement question. We show that given $f(x)$ positive, concave, increasing to ∞ on $(0, \infty)$ with $f(x + 1) - f(x) \to 0$ as $x \to \infty$ and

$$\limsup_{n\to\infty} \frac{f(n)}{A_n} \leqq 1,$$

there is a permutation π with $\sum_1^n a_{\pi(k)} \sim f(n)$. (Compare Riemann's classical theorem on rearrangement of conditionally convergent series [5]).

In this paper, all series are to be series of positive real numbers with

Received by the editors on April 20, 1981, and in revised form on July 20, 1981.