ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 13, Number 1, Winter 1983

DIFFERENTIAL EQUATIONS INVOLVING CIRCULANT MATRICES

ALAN C. WILDE

1. Introduction. This paper develops a theory for the solution of ordinary and partial differential equations whose structure involves the algebra of circulants. Recent interest of circulants is evident in a book by Davis [1]. This paper shows how the algebra of 2×2 circulants relates to the study of the harmonic oscillator, the Cauchy-Riemann equations, Laplace's equation, the Lorentz transformation, and the wave equation. It then uses $n \times n$ circulants to suggest natural generalizations of these equations to higher dimensions.

2. The algebra of circulants. An $n \times n$ circulant is a matrix of the form

$$X = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & \cdots & x_{n-2} & x_{n-1} \\ x_{n-1} & x_0 & x_1 & x_2 & \cdots & x_{n-3} & x_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ x_2 & x_3 & x_4 & x_5 & \cdots & x_0 & x_1 \\ x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-1} & x_0 \end{bmatrix}.$$

Note that X has arbitrary entries $x_0, x_1, \ldots, x_{n-1}$ in the top row and the entries are moved over one place to the right in each succeeding row. Let K denote the circulant with $x_1 = 1$ and $x_j = 0$ for all $j \neq 1$. Then the arbitrary circulant X equals $\sum_{h=0}^{n-1} x_h K^h$, and $K^n = I$. [$K^0 = I$ also.]

Define complex circulants $E_0, E_1, \ldots, E_{n-1}$ by

$$\text{(1)} \qquad E_h = (1/n) \sum_{j=0}^{n-1} \zeta^{-hj} K^j \text{ for } 0 \leqq h \leqq n-1,$$

where $\zeta = e^{2\pi i/n}$. Then $\{E_0, E_1, \ldots, E_{n-1}\}$ is an idempotent basis for complex circulants since

$$\text{(2.1)} \qquad E_h^2 = E_h \text{ for } 0 \leqq h \leqq n-1;$$

$$\text{(2.2)} \qquad E_h E_j = 0 \text{ if } h \neq j; \text{ and}$$

$$\text{(2.3)} \qquad E_0 + E_1 + \cdots + E_{n-1} = I. \text{ (See Davis [1]).}$$

Received by the editors on June 1, 1981.