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DIFFERENTIAL INEQUALITY TECHNIQUES AND SINGULAR PERTURBATIONS

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. The use of differential inequality techniques in the study of singular perturbation problems for ordinary and partial differential equations has a short but interesting history. In this paper we delineate briefly several avenues of investigation, starting from the original work and following its influence up to the present.

2. The work of Nagumo. In the late 1930's the Japanese mathematician M. Nagumo wrote two beautiful papers on differential inequalities, one concerned with two-point boundary value problems and the other specifically with a singularly perturbed initial value problem. These papers comprise the opening chapter of our story, and so let us spend a little time describing their contents.

The first paper of Nagumo [**29**] (cf. also [**21**]) concerns the existence of solutions of the Dirichlet problem

$$(2.1)\qquad \begin{aligned} &y'' = f(t, y, y'),\ a < t < b,\\ &y(a) = A,\ y(b) = B, \end{aligned}$$

where f is a continuous function on $[a, b] \times \mathbf{R}^2$. Under the assumptions that f grows at most quadratically with respect to y' (that is, $f(t, y, z) = O(|z|^2)$ as $|z| \to \infty$ for (t, y) in bounded subsets of $[a, b] \times \mathbf{R}$) and that there exists a $C^{(2)}$-bounding pair of functions $\{\alpha, \beta\}$ for the problem (2.1) (that is, functions α and β of class $C^{(2)}[a, b]$ satisfying $\alpha \leqq \beta$, $\alpha(a) \leqq A \leqq \beta(a)$, $\alpha(b) \leqq B \leqq \beta(b)$, and for t in (a, b), $\alpha'' \geqq f(t, \alpha, \alpha')$, $\beta'' \leqq f(t, \beta, \beta')$), Nagumo showed the existence of a $C^{(2)}$-solution $y = y(t)$of (2.1) satisfying $\alpha(t) \leqq y(t) \leqq \beta(t)$ in $[a, b]$. Thus he was not only able to prove that a solution exists, but also to give an estimate for this solu-

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