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CONDITIONS FOR COUNTABILITY OF THE SPECTRUM OF A SEPARABLE C*-ALGEBRA

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Most of the notation will be taken from [**4**]. A will always denote a separable C^*-algebra with spectrum $\hat{A}$ [**4**; p. 9]. We shall characterize those A with countable spectrum. Other characterizations can be found in [**1**; p. 292] and [**7**]. The theorem in this paper answers a question raised by Vaughn Jones.

If f is any state of A, we may define a semi-norm $\|\cdot\|_f$ on A by $\|a\|_f = f(a^*a)^{1/2}$. The state f is faithful exactly when $\|\cdot\|_f$ is a norm. For each equivalence class $\{\phi\}$ in $\hat{A}$ we select a representative ϕ and let H_ϕ be the representation space of ϕ. Since A is separable, so is H_ϕ. Let $\{\eta_{\phi n}\}_{n=1}^\infty$ be an orthonormal basis in H_ϕ. There is a unique, minimal central projection x_ϕ in the second dual A'' of A such that $x_\phi A''$ is isomorphic to the weak closure $\phi(A)''$ of $\phi(A)$ on the space H_ϕ [**4**; sect. 3.8]. Let Q denote the quasi-state space of A [**4**; p. 44] and P the set of pure states of A [**4**; p. 69]. Q is compact, convex and metrizable with the weak* topology, and $P \cup \{0\}$ is the set of extreme points of Q. By Choquet's theorem [**5**; p. 19], for each $f \in Q$ there is a representing measure m on the Borel subsets of Q, supported within P, such that for any $a \in A$, $\int_Q g(a)\, dm(g) = f(a)$. For any central projection x_ϕ as above let $P_\phi = \{g \in P: g(x_\phi) = 1\}$. For any unit vector $\eta \in H_\phi$, the pure state $g_{\phi\eta}$ on A given by $g_{\phi\eta}(a) = \langle \phi(a)\eta, \eta\rangle$ has a support projection [**6**; p. 31] $p_{\phi\eta} \leqq x_\phi$ which is a 1-dimensional projection in A'' [**4**; 3.13.6]. By [**4**; 3.11.9] there is a sequence $\{a_n\} \subset A$ such that $\|a_n\| \leqq 1$ and $a_n \to (1 - p_{\phi\eta})$ strongly in A''. Since each a_n, considered as a function on Q (see [**4**; p. 69]), is continuous, hence measurable, $p_{\phi\eta}$ is also measurable. Since the series $\sum_{n=1}^\infty p_{\phi\eta}$ converges pointwise on Q to the function represented by x_ϕ, then x_ϕ is also a measurable function on Q. Consequently P_ϕ is a measurable set for every $\phi \varepsilon \hat{A}$.

THEOREM. *The following are equivalent.*

(1) *$\hat{A}$ is countable.*

(2) *There is a faithful state f on A such that, for any proper C^*-subalgebra B of A, the $\|\cdot\|_f$ closure of any bounded ball in B does not contain the unit ball of A.*

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