ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 12, Number 3, Summer 1982

EXTENSIONS OF ℓ-HOMOMORPHISMS

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ABSTRACT. It is shown that, in abelian ℓ-groups, each morphism to a complete vector lattice extends over any majorizing embedding. This extends a result of the first author for Archimedean f-algebras with identity, and the recent Luxemburg-Schep theorem for vector lattices, and solves a problem of Conrad and McAlister. The proof presented here differs substantially from the Luxemburg-Schep proof. Ours uses the Yosida representation and Gleason's theorem on topological projectivity—this is novel, and seems relatively economical and transparent. The ℓ-group theorem is shown to imply, and with some modestly categorical machinery, to be implied by, certain similar statements in subcategories of ℓ-groups.

1. Introduction. Recall that, in a category $\mathscr{C}$, an object V is called injective if given the morphisms $\phi : G \to V$ and $\mu : G \to H$, with μ monic, there is a morphism $\varepsilon : H \to V$ with $\varepsilon \circ \mu = \phi$. We consider the category of Archimedean ℓ-groups (i.e., lattice-ordered groups), with morphisms the ℓ-homomorphisms, (i.e., group homomorphisms preserving finite meets and joins). Here, there are no injectives [**4**], but the theorem of the abstract, stated precisely below, shows that the complete vector lattices behave like injectives with respect to a restricted class of monics.

THEOREM 1.1 *Let $\phi : G \to V$ and $\mu : G \to H$ be morphisms of Abelian ℓ-groups. Then, there is a morphism $\varepsilon : H \to V$ with $\varepsilon \circ \mu = \phi$ provided that*

(a) *V is a complete vector lattice, and*

(b) *μ is a majorizing embedding.*

Here, *complete* means Dedekind complete; *embedding* is another word for monic or one-to-one; the subset S of the ℓ-group H is said to *majorize* H if given $h \in H$ there is $s \in S$ with $|h| \leqq s$; and the morphism $\mu : G \to H$ is called *majorizing* if $\mu(G)$ majorizes H.

[1]This author is indebted to Dompier's HRS and Wesleyan University for support during a sabbatical of Spring 1980.

AMS [*MOS*] *subject classifications* [1980]: 06F20, 54H10.

Received by the editors on September 8, 1980, and in revised form on December 9, 1980.