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CHAIN CONDITIONS AND INTEGRAL EXTENSIONS

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ABSTRACT. Let R be a ring, integral of bounded degree n over a subring C of the center of R. Various theorems are proved relating chain conditions in R and C. When R is a semi-prime ring which is torsion free for the regular elements of C and which is $n!$-torsion free, then R is a Goldie ring when C is. If also, $n!$ is a unit in R and C is "integrally closed" then C is a Noetherian ring (has Krull dimension) if and only if R is a Noetherian ring (has Krull dimension). Finally, assuming only that $n!$ is a unit in R, any R module has Gabriel dimension exactly when it does as a C module, in which case the dimensions are equal.

In this paper we consider rings which are integral extensions of central subrings and investigate whether certain chain conditions on one ring of the pair transfer to the other. Specifically, we examine the situation when one of the rings is a Goldie ring, a Noetherian ring, or has Krull dimension, and also consider modules over the larger ring which have Gabriel dimension with respect to one of the rings. To show that these conditions transfer from one ring to the other, we need to assume that the degrees of integrality are bounded, and usually, that this bound is invertible. Examples are presented to show that these assumptions are necessary. For the case of Noetherian rings, or of rings with Krull dimension, we must also assume that the central subring is "integrally closed" in its quotient ring, although we do not know if this assumption is necessary.

The questions studied here were raised as a consequence of similar ones for rings with involution [**16** and **17**], where the situation of a ring quadratic over its center occurs as a special case which must be considered. The more specific subject of algebraic algebras of bounded degree has been studied in the past, but the work was concerned with problems of a different kind. Some results on the finite dimensionality of such algebras were obtained in [**12**] and [**15**], such as for division algebras over their centers [**12**; Theorem 7, p. 701], or for semi-simple algebras over infinite

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