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# INFINITE PERMUTABLE SUBGROUPS

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**1. Introduction.** Suppose $H$ is a core-free permutable subgroup of the group $G$. This means that $H$ contains no non-identity normal subgroup of $G$ and that $HK = KH$ for each subgroup $K$ in $G$. If $G$ is finite, then Itô and Szep [6] proved that $H$ must be nilpotent. This result was improved by Maier and Schmid [7] who showed that $H$ is contained in $Z_\infty(G)$, the hypercenter of $G$, when $G$ is finite. The motivation behind the present paper was to investigate what happens when $G$ is infinite.

It is known in general that $H$ must be residually a finite nilpotent group ([1] and [8]). This result seems less satisfying, however, when it is recalled that any free group is also residually a finite nilpotent group. Another approach to the structure of $H$ is to consider the subgroup of $H$ generated by all its elements of finite order. It follows from results in [2] that this subgroup, which I denote by $T(H)$, is both locally finite and locally nilpotent. It is natural then, to ask what can be said about $H/T(H)$. This question seemed even more pertinent when the author realized that in all the examples of core-free permutable subgroups previously known (to the author, at least), $H/T(H)$ is abelian. If it were true that $H/T(H)$ is locally nilpotent, then it would follow that $H$ is locally solvable.

It is shown in [1] and [8] how to construct examples in which $H$ is not nilpotent nor even solvable. These examples are constructed by taking the direct sum of groups of prime-power-order using infinitely many distinct primes. One consequence of the present paper is that even when $G$ is a $p$-group, $H$ need not be solvable. The major thrust of this paper, however, is to settle the question of whether $H$ need be locally nilpotent or locally solvable. We will do a little more than this by constructing an example in which $H/T(H)$ is not locally solvable.

As far as the result of Maier and Schmid is concerned, there are various natural ways to try to generalize this result to infinite groups. For example, one could work with ascending series and ask whether $H \leqq Z_\infty(G)$. Alternately, one could work with descending series and ask whether $[H, G; \infty]$ or $[G, H; \infty]$ is the identity. (This notation is explained in the next section.) The answers to all of these questions are no and the main result of this paper may be stated as follows.

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