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DIRICHLET SEMIGROUPS ON BOUNDED DOMAINS

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ABSTRACT. Let Γ be a simply connected domain in $\mathbf{R}^n$ with smooth boundary and let $d\mu = \rho(x)dx$ be a probability measure on Γ such that $0 < \varepsilon \leqq \rho(x) \leqq R < \infty$ a.e. on Γ. Let $a(x)$ be an $n \times n$ matrix valued function on Γ which is uniformly bounded and which is uniformly bounded below by $\lambda > 0$. It is shown that the maximal and minimal Dirichlet forms associated with $a(x)$ and $\rho(x)$ are represented by self-adjoint operators, each of which generates a hypercontractive semigroup.

1. Introduction. Suppose that $d\mu = \rho(x)dx$ is a probability measure on a domain $\Gamma \subset \mathbf{R}^n$ and that for $x \in \Gamma$, $a(x)$ is a positive definite $n \times n$ matrix. We can define a sesquilinear form h on $L^2(\Gamma, d\mu)$ by specifying

$$h(u, v) = \int_\Gamma \nabla u \cdot (a\nabla \bar{v})dx = \sum_{i,j=1}^{n} \int_\Gamma u_{x_i}\bar{v}_{x_j}a_{ij}dx$$

for some suitably chosen domain of h. If the boundary of Γ is sufficiently regular, if a and ρ satisfy certain boundedness conditions, and if the domain of h is chosen properly, then h will be a Dirichlet (i.e. closed Markov) form, as defined by Beurling and Deny **[2]** and by Fukushima **[5]**. h is then represented by a positive self-adjoint operator A which generates a sub-Markov semigroup of operators $\{e^{-tA}\}$ on $L^2(\Gamma, d\mu)$. It follows that $\{e^{-tA}\}$ is an L^p-contractive semigroup as will be seen in §3.

In the case $\Gamma = \mathbf{R}^n$ and $a(x) = \rho(x)I$, there has been a great deal of interest in the relationship between properties of the density ρ and hypercontractivity of the semigroup e^{-tA} (see, e.g., **[3]**, **[4]**, **[6]**, **[7]**, **[11]**, **[16]** and **[17]**). Conditions on ρ which have been shown to imply hypercontractivity fall generally into two classes: (i) conditions which stipulate the decay of $\rho(x)$ for large $|x|$; (ii) conditions pertaining to the regularity of ρ.

For the semigroup e^{-tA} to be hypercontractive, it is in fact necessary that the density $\rho(x)$ decay in some uniform sense like $e^{-|x|^2}$. The regularity

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