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HOLOMORPHIC FUNCTIONS COMMUTING WITH ABSOLUTE VALUES

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Introduction. It is often possble in complex analysis to derive very strong conclusions about holomorphic functions from apparently weak information. Suppose, for example, that f is holomorphic in a disk centered at 0 in the complex plane, and that f commutes with absolute values in the sense that

$$f(|z|) = |f(z)| \tag{1}$$

One can then conclude that

(2) $f(z) = cz^m$, where $c \geqq 0$, and m is a non-negative integer.

A proof of this exercise usually relies on a power series expansion for f. In this note we extend this result in two directions. First of all, we observe that if Ω is a simply connected domain, not containing 0, and such that (1) makes sense for all z in Ω, then we can conclude that

(3) $f(z) = cz^\alpha$ where $c \geqq 0$, and α is an arbitrary real number.

Secondly, if Ω is a domain in $\mathbf{C}^n$ for which real powers of z are holomorphic, and $|z| = (|z_1|, |z_2|, \ldots, |z_n|)$, we can still conclude that (3) holds, except α is then an arbitrary real multi-index.

Our proof relies on the polar form of the Cauchy-Riemann equations, and integration of some real ordinary differential equations.

Statement and proof of the result. Let Ω be an open domain in $\mathbf{C}^n$. We say Ω is R-like if whenever z lies in Ω, so does $|z|$, Here $|z| = (|z_1|, |z_2|, \ldots, |z_n|)$. We say Ω is L-like if the functions $g(z) = \log(z_j)$ are all holomorphic on Ω. In particular this implies that Ω does not intersect any of the coordinate axes. Furthermore, if Ω is L-like, the function $g(z) = z^\alpha$ is holomorphic for any real multi-index α. Note that both concepts, L-like and R-like, are not preserved under general holomorphic changes of coordinates, because the origin and the notion of absolute value must remain invariant.

We recall that f is holomorphic on Ω if any only if f is continuously dif-

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