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# COMMUTATORS, ANTI-COMMUTATORS AND EULERIAN CALCULUS

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ABSTRACT. Consider operators obeying the commutation rule $Cy = 1 + qyC$ generalizing the rule $Dx = 1 + xD$, $D$ denoting $d/dx$. The case $q = 1$ corresponds to boson creation-annihilation operators, $q = -1$ to fermion operators. We derive Leibniz' rule for general $q$. We find that the canonical representation of $C$ such that $C1 = 0$ is given by the Eulerian derivative. The basics of Eulerian calculus are discussed and an indication of a discrete Hamiltonian theory analogous to the Heisenberg representation in quantum mechanics is given.

**Introduction.** In quantum theory one encounters creation and annihilation operators of two basic types: "boson" operators $a$, $b$ such that $ab - ba = 1$ and "fermion" operators $A$, $B$ such that $AB + BA = 1$. It is fairly easy to check, as will be seen later, that a standard representation of the pair $(a, b)$ is $a = d/dx$, $b = x$ acting on a space of functions $f(x)$. Our approach is to find a general explicit formula for calculating commutators of the form $[h(a), g(b)]$ for fairly general $h$ and $g$, e.g., any two polynomials. This is a generalization of Leibniz' rule for differentiating a product of functions. In finding a general Leibniz rule and, hence, an operator calculus for fermion operators it is natural to consider the general case of operators $\alpha$, $\beta$ such that $\alpha\beta = 1 + q\beta\alpha$ where $q$ is a fixed parameter, $q = 1$ for bosons, $q = -1$ for fermions. Then the operator calculus turns out to be the "$q$-calculus" or, following G.C. Rota, "Eulerian" calculus. This calculus arises quite naturally in the study of elliptic functions and generalized hypergeometric functions. This paper provides another natural setting for the Eulerian calculus.

We first review the approach for the boson or Heisenberg case and then proceed to the general, Eulerian, case. We conclude with an Eulerian, discrete, analog of Hamiltonian theory corresponding to the Heisenberg representation in quantum mechanics.

I. **Preliminary discussion.** The main result of this paper is Leibniz' rule for the functional calculus of the "Eulerian derivative"

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