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A NONSTANDARD PROOF OF THE MARTINGALE CONVERGENCE THEOREM

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In this note we use A. Robinson's [5] nonstandard analysis to give an elementary proof of the almost everywhere convergence of an L^1-bounded submartingale. Here, the index set $\mathscr{I}$ is a countable subset of the real numbers $\mathbf{R}$; we assume that $\mathscr{I}$ contains the natural numbers $\mathbf{N}$, but any cofinal subset of $\mathbf{R}$ will do. The continuous parameter martingale convergence theorem usually reduces to the case considered here. Our proof does not use the notion of a stopping time. It does employ a nonstandard criterion for almost everywhere convergence and demonstrates the usefulness of that criterion. It also produces the limit function.

We shall use the notation from [4] to which we refer the reader for further details about nonstandard analysis in general. We assume that we are working with a fixed $\aleph_1$-saturated, nonstandard extension of a standard structure. Of course, $^*\mathbf{R}$ and $^*\mathbf{N}$ denote the nonstandard extensions of $\mathbf{R}$ and $\mathbf{N}$, and $a \cong b$ means that $a - b$ is infinitesimal in $^*\mathbf{R}$. If $(X, \mathscr{F}, \mu)$ is an internal measure space and $g: X \to {}^*\mathbf{R} \cup \{-\infty, +\infty\}$ is internal and $\mathscr{F}$-measurable, then (following K. Stroyan) we shall say that $g \cong 0$ nearly surely (n.s.) when the following holds: For some infinitesimal $\varepsilon > 0$, $\mu(|g| > \varepsilon) \cong 0$. Clearly, $g \cong 0$ n.s. if and only if for each $\varepsilon > 0$ in $\mathbf{R}$, $\mu(|g| > \varepsilon) < \varepsilon$.

We now establish a nonstandard criterion for almost everywhere convergence. Here, as later, $\mathscr{I}$ denotes a countable subset of $\mathbf{R}$ with $\mathbf{N} \subset \mathscr{I}$. The ordering on $\mathscr{I}$ is the ordering inherited from $\mathbf{R}$. We shall use n, m, and k to denote natural numbers, while i and j will denote elements of $\mathscr{I}$ or $^*\mathscr{I}$. Moreover, $\{i: n \leqq i \leqq m\}$ will denote the set of indices in just $\mathscr{I}$ with $n \leqq i \leqq m$, while if γ and η are in $^*\mathbf{N} - \mathbf{N}$, then $\{i: \gamma \leqq i \leqq \eta\}$ will denote the set of indices in $^*\mathscr{I}$ with $\gamma \leqq i \leqq \eta$. Given $n \in \mathbf{N}$, $\bigcup_{i \geqq n} A_i$ will denote $\bigcup \{A_i: i \in \mathscr{I}, i \geqq n\}$.

THEOREM 1. *Let $(X, \mathscr{F}, \mu)$ be a standard measure space with $\mu(X) < +\infty$, and for each $i \in \mathscr{I}$, let g_i be an extended real-valued, $\mathscr{F}$-measurable function on X.*

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