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A SIMPLE PROOF AND GENERALIZATION OF WEGLORZ' CHARACTERIZATION OF NORMALITY FOR IDEALS

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ABSTRACT. A condition equivalent to normality for κ-complete ideals on a regular uncountable cardinal κ has been established by B. Weglorz as a corollary to his study of Ramsey and pseudonormal ideals. By isolating a critical combinatorial property (see Lemma 3) we are able to provide a direct, elementary proof of this equivalence and to generalize the result to arbitrary non-principal ideals.

1. Notation and definitions. Our notation is that used in Baumgartner, Taylor, Wagon [1]. If κ is a regular uncountable cardinal, an *ideal on* κ is a collection I of subsets of κ such that whenever $X, Y \in I$ and $Z \subseteqq X \cup Y$, then $Z \in I$. I is called *non-principal* if I contains all the singleton subsets. I is called *proper* if $\kappa \notin I$. I is κ*-complete* if whenever $\beta < \kappa$ and $\{X_\alpha | \alpha < \beta\} \subseteqq I$, then $\bigcup_{\alpha<\beta} X_\alpha \in I$. An important ideal on κ is the generalized Fréchet ideal, $I_\kappa = \{X \subseteqq \kappa | \, |X| < \kappa\}$. Note that if I is a non-principal, κ-complete ideal on κ, then $I_\kappa \subseteqq I$. However, we do not wish to restrict our attention in this paper to κ-complete ideals; the phrase "I is an (arbitrary) ideal on κ" will simply mean "I is a proper, non-principal ideal on κ".

If I is an ideal on κ, then $I^+ = \{X \subseteqq \kappa \mid X \notin I\}$ and $I^* = \{X \subseteqq \kappa | \kappa - X \in I\}$. Sets in I are said to be of "I-measure zero", sets in I^+ are said to be of "positive I-measure", and sets in I^* are said to be of "I-measure one."

If I is an ideal on κ and $A \in I^+$, then the *restriction of I to A*, denoted by $I|A$, is the ideal on κ given by $I|A = \{X \subseteqq \kappa | X \cap A \in I\}$.

If I is an ideal on κ and $A \subseteqq \kappa$ and f: $A \to \kappa$ is a function, f is called *I-small* if and only if for every $\alpha < \kappa, f^{-1}(\{\alpha\}) \in I$; f is called *regressive on* A if and only if for every $\alpha \in A - \{0\}, f(\alpha) < \alpha$.

If $\{X_\alpha | \alpha < \kappa\}$ is a sequence of κ-many subsets of κ, then the *diagonal union* of the sequence, denoted by $\nabla\{X_\alpha | \alpha < \kappa\}$ or by $\nabla_{\alpha<\kappa} X_\alpha$, is defined to be $\{\beta < \kappa | \exists \alpha < \beta, \beta \in X_\alpha\} = \bigcup\{X_\alpha - (\alpha + 1) \mid \alpha < \kappa\}$.

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