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CRYSTALLOGRAPHIC GROUPS AND THEIR MATHEMATICS

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Introduction. Symmetry, whether found or created, seems to have been a major interest of human beings for thousands of years. When Galois' insights were finally understood by the mathematical community, a precise and fruitful interpretation of symmetry began to play a central role in modern mathematics. One analyzes a gadget by looking for its symmetry; one describes the symmetry by studying transformations natural to the class of gadgets to which this particular one belongs and by isolating those transformations which preserve this symmetry.

The relationship between the every-day notion of symmetry and the theory of groups is lucidly presented in Weyl's classic, *Symmetry* [**4**]. Indeed, any introductory text on abstract algebra is remiss if it does not discuss the dihedral group (usually as the group of symmetries of a regular polygon). Many books on recreational mathematics go one step further and present the classification of "wallpaper designs", the plane crystallographic groups. We can, in fact, thank M. C. Escher for renewed attention to these matters ([**27**], [**36**], [**32**]). Yet it is very difficult for the more mature student to find an exposition of the general theory of n-dimensional crystallographic groups.

The features of an ideal mathematical crystal are easy to describe. One has a pattern that fills up n-space. The pattern can be reconstructed from a small, bounded piece by rigidly moving the piece around space. This is done so that the resulting pattern is "evenly spaced". The challenge is to translate such intuitive notions to actual mathematics.

It should be no surprise that once the groups of symmetries of a crystal were carefully studied, far-flung applications were found. In the past few years, algebraic topologists and differential geometers ([**15**], [**11**], [**10**]) have found these groups useful. Properties are discovered and rediscovered. A scorecard is needed. While these notes are not meant to provide a survey of all theorems using the words "crystallographic group", we hope that the foundations of the subject are pretty much here. We recommend Milnor's article in [**28**] as an introduction to this introduction. (Some of the material in these notes can also be found in

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