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A CONSTRUCTIVE PROXIMINALITY PROPERTY OF FINITE-DIMENSIONAL LINEAR SUBSPACES

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We recall that a subspace X of a metric space E is *proximinal* (in E) if, to each element in E, there corresponds a best approximant in X. Classically, every finite-dimensional linear subspace X of a normed linear space E is proximinal [5, Ch. 1, p. 20]. Our interest in a constructive development of best approximation theory was first aroused by the observation that all known proofs of this last proposition are nonconstructive. In fact, as we pointed out in §1 and Proposition 2.1 of [3], the constructive content of these proofs is the computability of dist(a, X) for each a in E. The further assertion that there exists a best approximant of a in X seems to depend on the attainment of the infimum of a continuous, real-valued function on a compact space, an essentially nonconstructive property of such functions (cf. [6, pp. 115-116]).

As this state of affairs appears to compromise the position of classical approximation theory as the foundation of a vast and powerful branch of numerical analysis, we believe that the systematic redevelopment of the classical theory along constructive lines provides a worthwhile and interesting mathmatical activity. We began this activity in [3], where we derived a result [3, Theorem 2.2] which was strong enough to yield a constructive proof of existence of minimax polynomial approximants of elements of $C[0, 1]$. Unfortunately, that result is not strong enough to cover other situations, such as that of best uniform approximation by linear combinations of functions in a general Tchebychev set, in which the classical theory proves existence and uniqueness of best approximants. In this paper, we present a constructive theorem which certainly disposes of the general Tchebychev approximation problem, and may well enable us to handle other unique existence problems in the bargain.

For general background material on constructive analysis, we refer to [2]. (A fuller, but less up-to-date, exposition of constructive mathematics is to be found in Bishop's fundamental treatise [1].) For our present purposes, an appreciation of the following facts and definitions is certainly necessary.

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