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THE HAM SANDWICH THEOREM AND SOME RELATED RESULTS

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ABSTRACT. Using integral transforms, a new and elementary proof of the ham sandwich theorem is presented. The proof requires a corollary of the Borsuk-Ulam theorem. Conversely, it is shown that the ham sandwich theorem implies this corollary. In the course of establishing the converse implication, a weak L^1 inversion theorem for the Radon transform is obtained.

It is our purpose in this paper to establish an interrelation between two known results in topology and geometry. The technique of proof of our theorems involves the theory of integral transforms which is of considerable independent interest.

Given a set E in n-dimensional Euclidean space $\mathbf{R}^n$, we shall call any positive measure μ with support contained in E a *mass density* of E. A measure, μ, not assumed to be necessarily positive, will be called a *signed mass density*. We shall assume throughout that the mass densities have zero absolute mass on any hyperplane of $\mathbf{R}^n$. Given any measure μ whose total mass on $\mathbf{R}^n$ satisfies

$$(1) \qquad \int_{\mathbf{R}^N} d\mu(x) \neq 0,$$

it is evident that a hyperplane defined by $\{x \in \mathbf{R}^n | \langle \xi, x \rangle = p\}$ may be chosen which divides the mass of E into two equal parts. Indeed, given any $\xi \in \mathbf{R}^n$, $\xi \neq 0$, the mass of the set

$$(2) \qquad E \cap \{x \in \mathbf{R}^n | \langle \xi, x \rangle \leqq p\}$$

is a continuous monotonic function of p. For each fixed ξ, the mass of the set described by (2) approaches either zero or the value of the integral in (1) as $|p| \to \infty$. Thus, the division of the mass of E, as defined by (1), into two equal parts is a simple consequence of the intermediate value theorem. Since no restriction has been made on the value of ξ, it is reasonable to conjecture that n sets $E_1, \ldots, E_n$ with mass densities satisfying (1) can be simultaneously divided into two equal parts by a fixed hyperplane. Actually, a somewhat stronger assertion can be shown which we now state for later reference.

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